Between Analytic Combinatorics and Species Theory

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Carine Pivoteau, joint work with Bruno Salvy and Michèle Soria

I Introduction

Species of structures

Well-founded combinatorial systems

Newton iteration

"Algorithms for combinatorial structures"



- Algorithms for analytic combinatorics:
 - Enumeration in Quasi-Optimal Complexity
 - Exponential and Ordinary Oracle
 + Differential Systems
- Well-defined input provided by species theory.

Bonus: Unified framework for constructible combinatorial classes.

Species of structures

classes \mathcal{E} (neutral) and \mathcal{Z} (atomic).

Well-founded combinatorial systems

Newton iteration

Symbolic Method: combinatorial specifications

Definition 1.7. A specification for an r-tuple $A = (A^{(1)}, ..., A^{(r)})$ of classes is a collection of r equations, $\begin{pmatrix} A^{(1)} = \Phi_1(A^{(1)}, ..., A^{(r)}) \\ A^{(r)} = \Phi_2(A^{(1)}, ..., A^{(r)}) \\ A^{(r)} = \Phi_2(A^{(1)}, ..., A^{(r)}) \end{pmatrix}$ where each Φ_1 denotes a term built from the A using the constructions of disjoint union, carterism product, scenarec, onerset, multica, and vecle as well as the initial

Species of structures

Well-founded combinatorial systems

Newton iteration













Species of structures

Well-founded combinatorial systems

Newton iteration







Species of structures

Well-founded combinatorial systems

Newton iteration



Species of structures

Well-founded combinatorial systems

Newton iteration





 $y_2 = 1$



 $y_2 = 1$

II Elements of Species Theory



Theorem (Implicit Species Theorem - Joyal 81)

Let \mathcal{H} be a vector of multisort species, such that

- $\mathcal{H}(\mathbf{0},\mathbf{0})=\mathbf{0}$ and
- the Jacobian matrix $\partial \mathcal{H} / \partial \mathcal{Y}(\mathbf{0},\mathbf{0})$ is nilpotent.

The system of equations

$${\boldsymbol{\mathcal{Y}}} = {\boldsymbol{\mathcal{H}}}({\boldsymbol{\mathcal{Z}}},{\boldsymbol{\mathcal{Y}}})$$

admits a vector S of species solution such that $S(\mathbf{0}) = \mathbf{0}$, which is unique up to isomorphism.

• $\mathcal{H}(\mathbf{0},\mathbf{0}) = \mathbf{0}$?

- What about the other condition?
- Is it a well founded system?

Definition

Definition (Species \mathcal{F})

A species of structures \mathcal{F} is a **rule** that,

- \forall finite set *U* produces a finite set $\mathcal{F}[U]$, and
- \forall bij. $\sigma: U \to V$ produces a bij. $\mathcal{F}[\sigma]: \mathcal{F}[U] \to \mathcal{F}[V]$,

with $\mathcal{F}[\mathsf{Id}_U] = \mathsf{Id}_{\mathcal{F}[U]}$ and $\mathcal{F}[\tau \circ \sigma] = \mathcal{F}[\tau] \circ \mathcal{F}[\sigma]$ for any bijections σ and τ .



- the empty species 0, defined by $0[U] = \emptyset$, $\forall U$.
- the species 1 (the empty set) defined by

 $1[U] = \emptyset$ if $U \neq \emptyset$ and $1[\emptyset] = \{\emptyset\}$.

 \bullet the species ${\mathcal Z}$ of singletons, defined by

 $\mathcal{Z}[U] = \{U\}$ if |U| = 1 and $\mathcal{Z}[U] = \emptyset$ otherwise.

- The species SET of sets, defined by $SET[U] = \{U\}$.
- The species SEQ of sequences (or linear orders) defined by $SEQ[\emptyset] = \{\emptyset\}$ and for $U = \{u_1, \dots, u_n\} \neq \emptyset$, $SEQ[U] = \{(u_{\sigma(1)}, \dots, u_{\sigma(n)}) \mid \sigma \in \mathcal{P}_n\}.$
- The species CYC of of cycles, defined by CYC[Ø] = Ø and for U ≠ Ø, CYC[U] = {σ | σ ∈ P[U] is composed of a unique cycle}.

In all cases, the transport $\mathcal{F}[\sigma]$ is (obvious) carried out in the natural fashion...

- The species SEQ of sequences is defined by SEQ[Ø] = {Ø} and for U = {u₁,..., u_n} ≠ Ø, SEQ[U] = {(u_{σ(1)},..., u_{σ(n)}) | σ ∈ P_n}.
- $\bullet\,$ The species ${\cal P}$ of permutations is defined by

 $\mathcal{P}[U] = \{ \psi : U \to U \mid \forall v \in U, \exists ! u \in U, \psi(u) = v \}.$

Species of structures

res Well-founded combinatorial systems

Intro

Sequences and Permutations

Species of structures

Intro

- The species SEQ of **sequences** is defined by $\text{SEQ}[\emptyset] = \{\emptyset\}$ and for $U = \{u_1, \ldots, u_n\} \neq \emptyset$, $\text{SEQ}[U] = \{(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) \mid \sigma \in \mathcal{P}_n\}$.
- $\bullet\,$ The species ${\cal P}$ of permutations is defined by

 $\mathcal{P}[U] = \{ \psi : U \to U \mid \forall v \in U, \exists ! u \in U, \psi(u) = v \}.$

Let $U = \{1, 2, 3\}$, $V = \{1, 2, 3\}$ and $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$, • SEQ[U] = {(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)} • $\mathcal{P}[U] = \left\{ \begin{array}{cccccc} 1 \to 1 & 1 \to 1 & 1 \to 2 & 1 \to 2 & 1 \to 3 & 1 \to 3 \\ 2 \to 2 & 2 & 2 \to 3 & 2 \to 1 & 2 \to 3 & 2 \to 1 & 2 \to 2 \\ 3 \to 3 & 3 \to 2 & 3 \to 3 & 3 \to 1 & 3 \to 2 & 3 \to 1 \end{array} \right\}$ • SEQ[σ] = {(1,2,3) \rightarrow (2,3,1), (1,3,2) \rightarrow (2,1,3),...} • $\mathcal{P}[\sigma] = \left\{ \begin{array}{ccccc} 1 \rightarrow 1 & 2 \rightarrow 2 & 1 \rightarrow 1 & 2 \rightarrow 2 \\ 2 \rightarrow 2 & \rightarrow & 3 \rightarrow 3 & , 2 \rightarrow 3 & \rightarrow & 3 \rightarrow 1 \\ 3 \rightarrow 3 & 1 \rightarrow 1 & 3 \rightarrow 2 & 1 \rightarrow 3 \end{array} \right\}$



• The Sum of species:

 $(\mathcal{F}+\mathcal{G})[U]=\mathcal{F}[U]+\mathcal{G}[U]$

• The **Product** of two species \mathcal{F} and \mathcal{G} :

$$(\mathcal{F} \cdot \mathcal{G})[U] = \sum_{(U_1, U_2), U = U_1 + U_2} \mathcal{F}[U_1] \times \mathcal{G}[U_2],$$

• The **Composition** of \mathcal{F} with \mathcal{G} (also denoted by $\mathcal{F}(\mathcal{G})$):





(2)

3)

2

, 1 • 2



What about the species of Permutations?

3





- Two species are equal if they produce the same sets and bijections.
- An isomorphism from *F* to *G* is a family of bijections α_U : *F*[U] → *G*[U] such that:



(2)

- !! isomorphic structures are a different notion !!
- = denotes a combinatorial equality between isomorphic species.

Combinatorial Systems

$$\begin{aligned} \boldsymbol{\mathcal{Y}} &= \boldsymbol{\mathcal{H}}(\boldsymbol{\mathcal{Z}},\boldsymbol{\mathcal{Y}}): \\ \begin{cases} \mathcal{Y}_1 &= \mathcal{H}_1(\boldsymbol{\mathcal{Z}},\mathcal{Y}_1,\mathcal{Y}_2,\ldots,\mathcal{Y}_m) \\ \mathcal{Y}_2 &= \mathcal{H}_2(\boldsymbol{\mathcal{Z}},\mathcal{Y}_1,\mathcal{Y}_2,\ldots,\mathcal{Y}_m) \\ \vdots \\ \mathcal{Y}_m &= \mathcal{H}_m(\boldsymbol{\mathcal{Z}},\mathcal{Y}_1,\mathcal{Y}_2,\ldots,\mathcal{Y}_m) \end{aligned}$$

multisort species: \forall finite sets U_1, \ldots, U_k , produces a set $\mathcal{F}[U_1, \ldots, U_k]$

Intro Species of structures Well-founded combinatorial systems Combinatorial Systems

$$\mathbf{\mathcal{Y}} = \mathcal{H}(\mathcal{Z}, \mathcal{Y}) :$$

$$\begin{cases}
\mathcal{Y}_{1} = \mathcal{H}_{1}(\mathcal{Z}, \mathcal{Y}_{1}, \mathcal{Y}_{2}, \dots, \mathcal{Y}_{m}) \\
\mathcal{Y}_{2} = \mathcal{H}_{2}(\mathcal{Z}, \mathcal{Y}_{1}, \mathcal{Y}_{2}, \dots, \mathcal{Y}_{m}) \\
\vdots \\
\mathcal{Y}_{m} = \mathcal{H}_{m}(\mathcal{Z}, \mathcal{Y}_{1}, \mathcal{Y}_{2}, \dots, \mathcal{Y}_{m}) \\
\mathcal{H}_{m} = \mathcal{H}_{m}(\mathcal{Z}, \mathcal{Y}_{m}) \\
\mathcal{H}_{m} = \mathcal{H}_{m}(\mathcal{H}, \mathcal{H}_{m}) \\$$

multisort species: \forall finite sets U_1, \ldots, U_k , produces a set $\mathcal{F}[U_1, \ldots, U_k]$

Newton iteration

Digression: generating series

Definition

Exponential generating series of \mathcal{F} : $F(z) = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!}$ Ordinary generating series of \mathcal{F} : $\tilde{F}(z) = \sum_{n=0}^{\infty} \tilde{f}_n z^n$.

Definition

Cycle index series of \mathcal{F} :

$$Z_{\mathcal{F}}(z_1, z_2, z_3, \dots) = \sum_{n \ge 0} \frac{1}{n!} \left(\sum_{\sigma \in \mathcal{P}_n} \operatorname{fix} \mathcal{F}[\sigma] z_1^{\sigma_1} z_2^{\sigma_2} \cdots \right), \quad (1)$$

where σ_i is the number of cycles of length *i* in the cycle decomposition of the permutation σ and fix $\mathcal{F}[\sigma]$ is the number of \mathcal{F} -structures on $\{1, \ldots, n\}$ fixed by $\mathcal{F}[\sigma]$.

$$F(z) = Z_{\mathcal{F}}(z, 0, 0, \dots)$$
 and $\tilde{F}(z) = Z_{\mathcal{F}}(z, z^2, z^3, \dots)$.

Derivative

Definition (Derivative)

The derivative \mathcal{F}' of a species \mathcal{F} is defined by $\mathcal{F}'[U] = \mathcal{F}[U + \{\star\}]$, where \star is an element chosen outside of U.



species	derivative	species	derivative
$\mathcal{A} + \mathcal{B}$	$\mathcal{A}' + \mathcal{B}'$	$\mathrm{Seq}(\mathcal{B})$	$\operatorname{Seq}(\mathcal{B}) \cdot \mathcal{B}' \cdot \operatorname{Seq}(\mathcal{B})$
$\mathcal{A} \cdot \mathcal{B}$	$\mathcal{A}'\cdot\mathcal{B}+\mathcal{A}\cdot\mathcal{B}'$	$\operatorname{Cyc}(\mathcal{B})$	$\operatorname{SeQ}(\mathcal{B})\cdot\mathcal{B}'$
		$\operatorname{Set}(\mathcal{B})$	$\operatorname{Set}(\mathcal{B})\cdot\mathcal{B}'$

Newton iteration

Jacobian matrix of ${m {\cal Y}}={m {\cal H}}({m {\cal Z}},{m {\cal Y}})$: $\partial {m {\cal H}}/\partial {m {\cal Y}}$

Example:

Intro

$$\mathcal{H}(\mathcal{G},\mathcal{S},\mathcal{P}):=\left(\mathcal{S}+\mathcal{P},\operatorname{Seq}(\mathcal{Z}+\mathcal{P}),\operatorname{Set}(\mathcal{Z}+\mathcal{S})
ight).$$

$$rac{\partial \mathcal{H}}{\partial \mathcal{Y}} = egin{pmatrix} 0 & 1 & 1 \ 0 & 0 & \operatorname{SEQ}(\mathcal{Z} + \mathcal{P}) \cdot 1 \cdot \operatorname{SEQ}(\mathcal{Z} + \mathcal{P}) \ 0 & \operatorname{SET}(\mathcal{Z} + \mathcal{S}) \cdot 1 & 0 \end{pmatrix}$$

Not nilpotent at (0, 0): successive powers

Species of structures

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

$$Rmk: \left. \left(\partial \mathcal{H} / \partial \mathcal{Y} \right)^k \right|_{(i,j)} = \frac{\partial \mathcal{H}_i}{\partial \mathcal{Y}_{\ell_1}} \frac{\partial \mathcal{H}_{\ell_1}}{\partial \mathcal{Y}_{\ell_2}} \cdots \frac{\partial \mathcal{H}_{\ell_k}}{\partial \mathcal{Y}_j}$$

Jacobian matrix of ${m {\cal Y}}={m {\cal H}}({m {\cal Z}},{m {\cal Y}})$: $\partial {m {\cal H}}/\partial {m {\cal Y}}$

Example:

Intro

$$\mathcal{H}(\mathcal{G}, \mathcal{S}, \mathcal{P}) := (\mathcal{S} + \mathcal{P}, \mathcal{Z} \cdot \operatorname{Seq}(\mathcal{Z} + \mathcal{P}), \operatorname{Set}(\mathcal{Z} + \mathcal{S})).$$

$$rac{\partial \mathcal{H}}{\partial \mathcal{Y}} = egin{pmatrix} 0 & 1 & 1 \ 0 & 0 & \operatorname{SEQ}(\mathcal{Z} + \mathcal{P}) \cdot \mathcal{Z} \cdot \operatorname{SEQ}(\mathcal{Z} + \mathcal{P}) \ 0 & \operatorname{SET}(\mathcal{Z} + \mathcal{S}) \cdot 1 & 0 \end{pmatrix}$$

Nilpotent at (0, 0): successive powers

Species of structures

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\operatorname{Rmk:} \left(\partial \mathcal{H} / \partial \mathcal{Y} \right)^k \Big|_{(i,j)} = \frac{\partial \mathcal{H}_i}{\partial \mathcal{Y}_{\ell_1}} \frac{\partial \mathcal{H}_{\ell_1}}{\partial \mathcal{Y}_{\ell_2}} \cdots \frac{\partial \mathcal{H}_{\ell_k}}{\partial \mathcal{Y}_j}$$

III Well-founded combinatorial systems

Back to Joyal's Implicit Species Theorem

Theorem (IST)

Species of structures

Intro

If $\mathcal{H}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ and $\partial \mathcal{H} / \partial \mathcal{Y}(\mathbf{0}, \mathbf{0})$ is nilpotent, then $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{Y})$ has a unique solution, limit of $\mathcal{Y}^{[0]} = \mathbf{0}, \qquad \mathcal{Y}^{[n+1]} = \mathcal{H}(\mathcal{Z}, \mathcal{Y}^{[n]}) \quad (n \ge 0).$

Def. $\mathcal{A} =_k \mathcal{B}$ if they coincide up to size k (contact k).

Key Lemma



The nature of combinatorial systems...

Species of structures

Joyal's Implicit Species Theorem is too restrictive:

- We don't want the condition $\mathcal{H}(\mathbf{0},\mathbf{0}) = \mathbf{0}$.
- To allow equations such as $\mathcal{Y} = 1 + \mathcal{Z}\mathcal{Y}$.
- We want to *characterize precisely* which are the systems that define combinatorial structures \triangleright well-founded systems.

Bonus :

Intro

A better understanding of the role played by the Jacobian matrix and a better knowledge of the structure of combinatorial systems.

General Implicit Species Theorem

Species of structures

Intro

Theorem (General Implicit Species Theorem)

Let $\mathcal{H} = (\mathcal{H}_{1:m})$ be any vector of species, such that the system $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{Y})$ is well-founded. This system admits a solution \mathcal{S} such that $\mathcal{S}(\mathbf{0}) = \mathcal{H}^m(\mathbf{0}, \mathbf{0})$, which is unique up to isomorphism.

Definition (Well-founded combinatorial system)

 $\boldsymbol{\mathcal{Y}}=\boldsymbol{\mathcal{H}}(\boldsymbol{\mathcal{Z}},\boldsymbol{\mathcal{Y}})$ is said to be well-founded when the iteration

$$\boldsymbol{\mathcal{Y}}^{[0]} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{\mathcal{Y}}^{[n+1]} = \boldsymbol{\mathcal{H}}(\boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{Y}}^{[n]}), \quad n \geq 0 \qquad (\Phi)$$

- is well-defined,
- defines a convergent sequence
- and the limit ${\cal S}$ of this sequence has no zero coordinate.



Joyal's conditions:

$$\begin{array}{lll} \mathcal{Y} = \operatorname{SEQ}(\mathcal{Z}) \checkmark & \mathcal{Y} = \operatorname{SEQ}(\mathcal{Z} \operatorname{SEQ}(\mathcal{Z})) \checkmark & \mathcal{Y} = \operatorname{SEQ}(\operatorname{SEQ}(\mathcal{Z})) \nearrow \\ \mathcal{H}'(0) = 0 & \mathcal{H}'(0) = 0 & \mathcal{H}'(0) \text{ not defined!} \\ \mathcal{Y} = \mathcal{Z} \mathcal{Y} \checkmark & \mathcal{Y} = \mathcal{Z} + \mathcal{Z} \mathcal{Y} \checkmark & \mathcal{Y} = \mathcal{Z} + \mathcal{Y} \checkmark \end{array}$$

$$\mathcal{H}'(0,0)=0$$

$$\mathcal{Y} = \mathcal{Z} + \mathcal{Z} \mathcal{Y} \checkmark$$
$$\mathcal{H}'(0,0) = 0$$

$$\mathcal{Y} = \mathcal{Z} + \mathcal{Y} \nearrow \mathcal{H}'(0,0) = 1$$



Joyal's conditions:

$$\begin{array}{l} \mathcal{Y} = \operatorname{SEQ}(\mathcal{Z}) \checkmark \quad \mathcal{Y} = \operatorname{SEQ}(\mathcal{Z} \operatorname{SEQ}(\mathcal{Z})) \checkmark \quad \mathcal{Y} = \operatorname{SEQ}(\operatorname{SEQ}(\mathcal{Z})) \checkmark \\ \mathcal{H}'(0) = 0 \qquad \qquad \mathcal{H}'(0) = 0 \qquad \qquad \mathcal{H}'(0) \text{ not defined!} \\ \end{array} \\ \begin{array}{l} \mathcal{Y} = \mathcal{Z} \mathcal{Y} \checkmark \qquad \mathcal{Y} = \mathcal{Z} + \mathcal{Z} \mathcal{Y} \checkmark \qquad \qquad \mathcal{Y} = \mathcal{Z} + \mathcal{Y} \checkmark \\ \mathcal{H}'(0,0) = 0 \qquad \qquad \mathcal{H}'(0,0) = 0 \qquad \qquad \mathcal{H}'(0,0) = 1 \end{array}$$

With our conditions:

 $\mathcal{Y} = \mathcal{Z} \mathcal{Y} \nearrow$ because $\mathcal{Y} = 0$.

How to detect 0 coordinates: Look for 0 in $\mathcal{H}^m(\mathcal{Z}, \mathbf{0})$.



Joyal's conditions:

$$\begin{array}{ll} \mathcal{Y} = \operatorname{SEQ}(\mathcal{Z}) \checkmark & \mathcal{Y} = \operatorname{SEQ}(\mathcal{Z} \operatorname{SEQ}(\mathcal{Z})) \checkmark & \mathcal{Y} = \operatorname{SEQ}(\operatorname{SEQ}(\mathcal{Z})) \nearrow \\ \mathcal{H}'(0) = 0 & \mathcal{H}'(0) = 0 & \mathcal{H}'(0) \text{ not defined!} \end{array}$$
$$\begin{array}{ll} \mathcal{Y} = \mathcal{Z} \mathcal{Y} \checkmark & \mathcal{Y} = \mathcal{Z} + \mathcal{Z} \mathcal{Y} \checkmark & \mathcal{Y} = \mathcal{Z} + \mathcal{Y} \checkmark \\ \mathcal{H}'(0,0) = 0 & \mathcal{H}'(0,0) = 0 & \mathcal{H}'(0,0) = 1 \end{array}$$

With our conditions:

Examples:

 $\mathcal{Y} = \mathcal{Z} \mathcal{Y} \nearrow$ because $\mathcal{Y} = 0$.

How to detect 0 coordinates: Look for 0 in $\mathcal{H}^m(\mathcal{Z}, \mathbf{0})$.

$$\begin{cases} \mathcal{A} = \mathcal{B} \\ \mathcal{B} = \mathcal{C} \\ \mathcal{C} = \mathcal{Z} \end{cases} \qquad \begin{cases} \mathcal{A} = \mathcal{B} \\ \mathcal{B} = \mathcal{Z} + \mathcal{C} \\ \mathcal{C} = \mathcal{Z} \mathcal{C} \end{cases}$$

Well founded at 0

Intro

Algorithm 0-coord: Detection of zero coordinates in the solution of a system

Input: $\mathcal{H} = (\mathcal{H}_{1:m})$: a vector of species such that $\mathcal{H}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ and the Jacobian matrix $\partial \mathcal{H} / \partial \mathcal{Y}(\mathbf{0}, \mathbf{0})$ is nilpotent.

Output: Answer to "Are there 0 coordinates in the solution of the system

$$\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{H}}(\boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{Y}})?$$
"

Species of structures

begin

Compute $\mathcal{U} := \mathcal{H}^m(\mathcal{Z}, \mathbf{0})$ foreach *coordinate* \mathcal{C} of \mathcal{U} do \sqcup if $\mathcal{C} = 0$ then return YES return NO end

Algorithm isWellFoundedAt0: Characterization of well-founded systems at 0

Input: $\mathcal{H} = (\mathcal{H}_{1:m})$: a vector of species such that $\mathcal{H}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. Output: Answer to "Is the system $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{Y})$ well-founded at $\mathbf{0}$?" begin Compute $\mathcal{J} := \partial \mathcal{H} / \partial \mathcal{Y}(\mathbf{0}, \mathbf{0})$ if $\mathcal{J}^m = 0$ then return $\mathbf{0}$ -coord(\mathcal{H}) else return NO end

Intro Species of structures Well-founded combinatorial systems Newton iteration $\mathcal{H}(\mathbf{0},\mathbf{0})=\mathbf{0}$ Examples $\begin{cases} \mathcal{Y}_1 = \mathcal{Z} \ \mathcal{Y}_2 \\ \mathcal{V}_2 = \mathcal{Z} \ \mathcal{Y}_1 \ \operatorname{SEQ}(\mathcal{Y}_2) \end{cases} \checkmark \begin{pmatrix} 0 & 0 \\ \mathcal{Z} \ \operatorname{SEQ}(\mathcal{Y}_2) & \mathcal{Z} \ \mathcal{Y}_1 \ \operatorname{SEQ}(\mathcal{Y}_2)^2 \end{pmatrix} \Big|_{0,0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{cases} \mathcal{Y}_1 = \mathcal{Z} + \mathcal{Y}_2 \\ \mathcal{V}_2 = \mathcal{Z} \mathcal{Y}_1 \operatorname{SEO}(\mathcal{Y}_2) \end{cases} \checkmark \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\begin{cases} \mathcal{Y}_1 = \mathcal{Z} + \mathcal{Y}_2 \\ \mathcal{Y}_2 = \mathcal{Z} + \mathcal{Y}_1 \operatorname{SEQ}(\mathcal{Y}_2) \end{cases} \land \begin{pmatrix} 0 & 1 \\ \operatorname{SEQ}(\mathcal{Y}_2) & \mathcal{Y}_1 \operatorname{SEQ}(\mathcal{Y}_2)^2 \end{pmatrix} \Big|_{0,0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\begin{cases} \mathcal{Y}_1 = \mathcal{Z} + \mathcal{Y}_2^2 \\ \mathcal{Y}_2 = \mathcal{Y}_1 \end{cases} \checkmark \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Algorithmic Characterization

Definition (Companion system of $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{Y})$)

The species 1 is replaced by the sort \mathcal{Z}_1 :

 $\boldsymbol{\mathcal{Y}}=\boldsymbol{\mathcal{K}}(\mathcal{Z}_1,\boldsymbol{\mathcal{Z}},\boldsymbol{\mathcal{Y}}), \quad \text{where} \quad \boldsymbol{\mathcal{K}}=\boldsymbol{\mathcal{H}}(\boldsymbol{\mathcal{Z}},\boldsymbol{\mathcal{Y}}){-}\boldsymbol{\mathcal{H}}(\boldsymbol{0},\boldsymbol{0}){+}\mathcal{Z}_1\boldsymbol{\mathcal{H}}(\boldsymbol{0},\boldsymbol{0}).$

Theorem (Characterization of well-founded systems)

The system $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{Y})$ is well-founded if and only if

- the companion system Y = K(Z₁, Z, Y) is well-founded at 0 (Joyal's conditions without zeroes)
- the solution S₁(Z₁, Z) of the companion system is polynomial in Z₁.

In this case, the limit of the iteration (Φ) is $S_1(1, \mathbb{Z})$.

Definition

 $\mathcal{F}(\mathcal{Z}_1, \mathcal{Z}_2)$ is polynomial in the sorts \mathcal{Z}_1 when, for all $n \ge 0$, the species $\mathcal{F}_{=(.,n)} = \sum_{k \ge 0} \mathcal{F}_{=(k,n)}$ is polynomial.

(The species is polynomial when the size is fixed in the other sorts.)

Examples:

- $SEQ(Z_1 + Z_2)$: not polynomial in Z_1 or Z_2
- $SEQ(Z_1 \cdot Z_2)$: polynomial in Z_1 and Z_2 (but not in Z)
- $\mathcal{Z}_1 SEQ(\mathcal{Z}_2)$: polynomial in \mathcal{Z}_1 and not in \mathcal{Z}_2 .

Well-founded Systems?

$$\begin{cases} \mathcal{Y}_1 = \mathcal{Z} + \mathcal{Y}_1 \mathcal{Y}_2 \\ \mathcal{Y}_2 = 1 \end{cases} \qquad \qquad \begin{cases} \mathcal{Y}_1 = \mathcal{Z} + \mathcal{Y}_2 \mathcal{Y}_1^2 \\ \mathcal{Y}_2 = 1 \end{cases}$$

Definition

 $\mathcal{F}(\mathcal{Z}_1, \mathcal{Z}_2)$ is polynomial in the sorts \mathcal{Z}_1 when, for all $n \ge 0$, the species $\mathcal{F}_{=(.,n)} = \sum_{k \ge 0} \mathcal{F}_{=(k,n)}$ is polynomial.

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- $\mathcal{Z}_1 SEQ(\mathcal{Z}_2)$: polynomial in \mathcal{Z}_1 and not in \mathcal{Z}_2 .

Well-founded Systems?

$$\begin{cases} \mathcal{Y}_1 = \mathcal{Z} + \mathcal{Y}_1 \mathcal{Y}_2 \\ \mathcal{Y}_2 = \mathcal{Z}_1 \end{cases}$$

$$\begin{cases} \mathcal{Y}_1 = \mathcal{Z} + \mathcal{Y}_2 \mathcal{Y}_1^2 \\ \mathcal{Y}_2 = \mathcal{Z}_1 \end{cases} \checkmark$$

Information given by the Jacobian Matrix

Role of the Jacobian Matrix (dependency graph of the system):

- **()** Well-founded systems at **0**: nilpotence of $\partial \mathcal{H} / \partial \mathcal{Y}(\mathbf{0}, \mathbf{0})$
- Implicit polynomial species: nilpotence of ∂H/∂Y(Z, Y) (detection of cycles in the graph)

$$\begin{array}{rcl} \mathcal{A} & = & \mathcal{Z} \times \mathcal{B} + \mathcal{C} \\ \mathcal{B} & = & \operatorname{SET}(\mathcal{A}) \\ \mathcal{C} & = & \mathcal{Z} + \operatorname{SEQ}(\mathcal{B}) \end{array} & \begin{pmatrix} 0 & \mathcal{Z} & 1 \\ \operatorname{SET}(\mathcal{A}) & 0 & 0 \\ 0 & \operatorname{SEQ}(\mathcal{B})^2 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ & & & & & & & \\ & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & \\ \end{array}$$

Newton iteration

Information given by the Jacobian Matrix

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- Implicit polynomial species: nilpotence of ∂H/∂Y(Z, Y) (detection of cycles in the graph)

• Implicit partially polynomial species: nilpotence of $\partial \mathcal{H} / \partial \mathcal{Y}(\mathcal{Z}_1, \mathbf{0}, \mathcal{S}(\mathcal{Z}_1, \mathbf{0}))$ (+ conditions on \mathcal{H} and $\mathcal{S}(\mathcal{Z}_1, \mathbf{0})$)

C

Newton iteration

Role of the Jacobian Matrix (dependency graph of the system):

- **()** Well-founded systems at **()**: nilpotence of $\partial \mathcal{H} / \partial \mathcal{Y}(\mathbf{0}, \mathbf{0})$
- Implicit polynomial species: nilpotence of ∂H/∂Y(Z, Y) (detection of cycles in the graph)
- Implicit partially polynomial species: nilpotence of \(\frac{\mathcal{H}}{\mathcal{D}\mathcal{V}}(\mathcal{Z}_1, 0, \mathcal{S}(\mathcal{Z}_1, 0)) \(+ \conditions on \(\mathcal{H}\) and \(\mathcal{S}(\mathcal{Z}_1, 0)\)\)
- Well-founded systems: both 1 and 3.
- The key for Newton iteration.

But no information on the 0 coord. (depend on initial conditions).

Well-founded combinatorial systems

Newton iteration

From Analytic Combinatorics to Species Theory

Analytic Combinatorics:

Intro

• Symbolic method to describe recursive combinatorial classes

Species of structures

- A restricted set of combinatorial constructions, with a dictionary for gfs.
- Powerful tools for enumeration (singularity analysis,...)
- Automatic random sampling (Recursive, Boltzmann)

But no information on well-founded systems...

From Analytic Combinatorics to Species Theory

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Species Theory:

- A more general framework for combinatorial structures
- Implicit species theorem
 Well-founded systems
- Combinatorial Derivative Labelle
- Newton iteration (Decoste, Labelle, Leroux)
- Differential systems

But no analytic tools...

IV Newton iteration

Between Analytic Combinatorics and Species Theory

Combinatorial Newton Iteration

Theorem (essentially Labelle)

For any well-founded system $\mathcal{Y} = \mathcal{H}(\mathcal{Z}, \mathcal{Y})$, if \mathcal{A} has contact k with its solution \mathcal{S} and $\mathcal{A} \subset \mathcal{H}(\mathcal{Z}, \mathcal{A})$, then

$$oldsymbol{\mathcal{A}} + \sum_{i\geq 0} \left(rac{\partial oldsymbol{\mathcal{H}}}{\partial oldsymbol{\mathcal{Y}}}(oldsymbol{\mathcal{Z}},oldsymbol{\mathcal{A}})
ight)^i \cdot (oldsymbol{\mathcal{H}}(oldsymbol{\mathcal{Z}},oldsymbol{\mathcal{A}})-oldsymbol{\mathcal{A}})$$

has contact 2k + 1 with S.



Quadratic Iteration

Intro

Theorem (essentially Labelle)

For any well-founded system, the sequence $\boldsymbol{\mathcal{Y}}^{[0]}=\boldsymbol{0}$ and

$$\boldsymbol{\mathcal{Y}}^{[n+1]} = \boldsymbol{\mathcal{Y}}^{[n]} + \left(\sum_{k \ge 0} \left(\frac{\partial \boldsymbol{\mathcal{H}}}{\partial \boldsymbol{\mathcal{Y}}}(\boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{Y}}^{[n]})\right)^k\right) \cdot \left(\boldsymbol{\mathcal{H}}\left(\boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{Y}}^{[n]}\right) - \boldsymbol{\mathcal{Y}}^{[n]}\right)$$

is well defined and converges quadratically to the solution ${\mathcal S}$ of the system.



Rmk : Generation by increasing Strahler numbers.

Between Analytic Combinatorics and Species Theory

Newton Iteration for Binary Trees

Species of structures

Intro

 $\mathcal{Y} = 1 + \mathcal{Z} imes \mathcal{Y}^2$

 $\mathcal{Y}_{n+1} = \mathcal{Y}_n + \operatorname{SEQ}(\mathcal{Z} \times \mathcal{Y}_n \times \star + \mathcal{Z} \times \star \times \mathcal{Y}_n) \times (1 + \mathcal{Z} \times \mathcal{Y}_n^2 - \mathcal{Y}_n).$

 Intro
 Species of structures
 Well-founded combinatorial systems

 Newton
 Iteration for
 Binary
 Trees

$$\mathcal{Y} = 1 + \mathcal{Z} imes \mathcal{Y}^2$$

$$\mathcal{Y}_{n+1} = \mathcal{Y}_n + \operatorname{SEQ}(\mathcal{Z} \times \mathcal{Y}_n \times \star + \mathcal{Z} \times \star \times \mathcal{Y}_n) \times (1 + \mathcal{Z} \times \mathcal{Y}_n^2 - \mathcal{Y}_n).$$

$$\mathcal{Y}_0 = 0$$
 $\mathcal{Y}_1 = \circ$





[Décoste, Labelle, Leroux 1982]