Fin de la classification des marches à petits pas dans le quart de plan

Stephen Melczer (SFU)
Joint work with Marni Mishna (SFU, LaBRI (2013))

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Given: A set of directions $\mathcal{S}$

Count: q_n = number of lattice walks in the first quadrant

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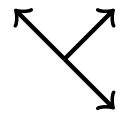

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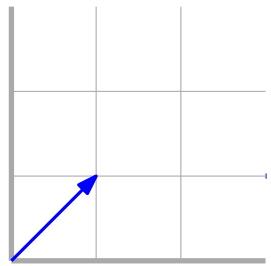

$$q_1 = 1$$

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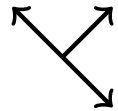

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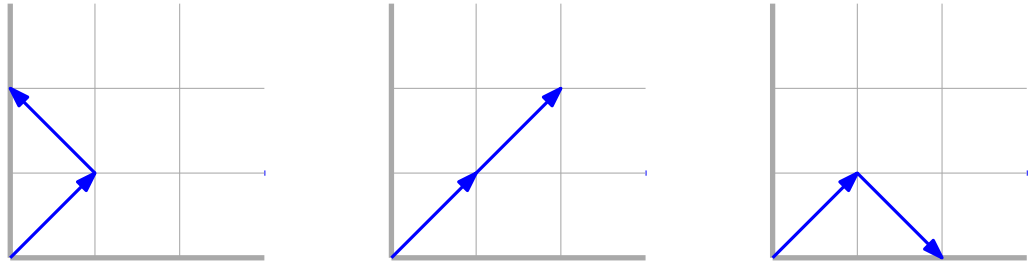

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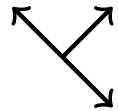

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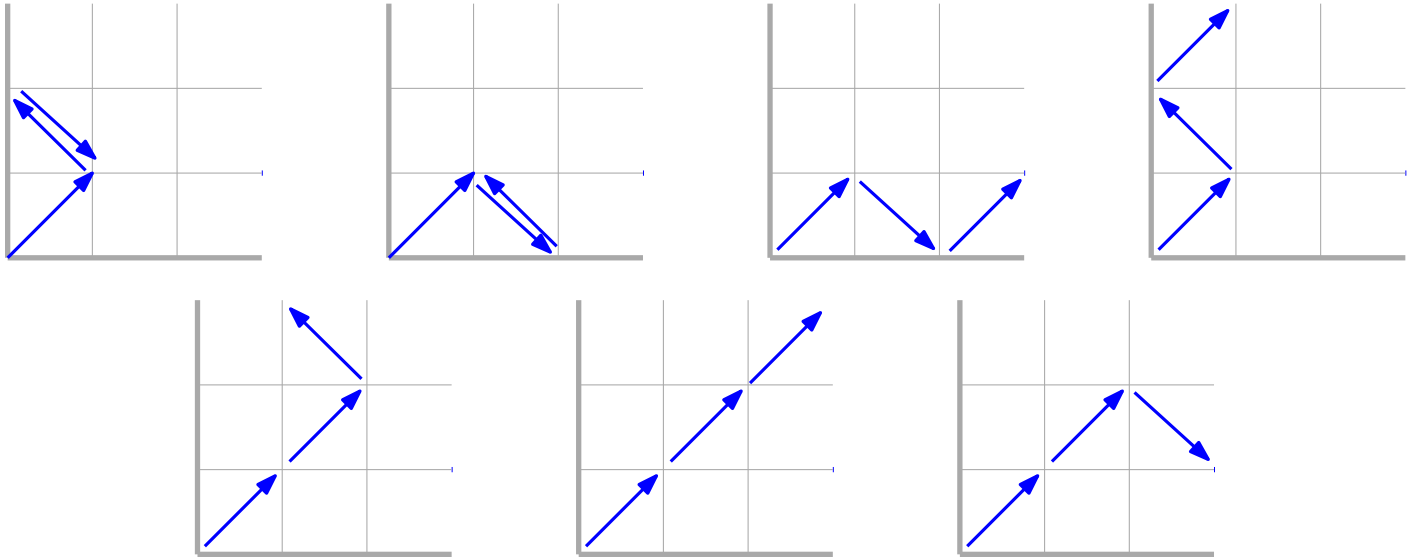

$q_3 = 7$

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These models are well suited to the iterated kernel method
[van Rensburg, Prellberg, and Rechnitzer 2008].

Context: Walks with Unit Steps

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Of $2^8 = 256$ step sets, there are 79 non-isomorphic 2D models [Bousquet-Mélou & Mishna 2010].

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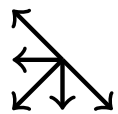

will never leave the origin.

Multivariate Generating Function

Tool: the generating function

$$Q(x, y; t) = \sum_{n,i,j \geq 0} q(i, j; n) x^i y^j t^n$$

which marks the number of walks of length n ending at (i, j).

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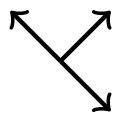
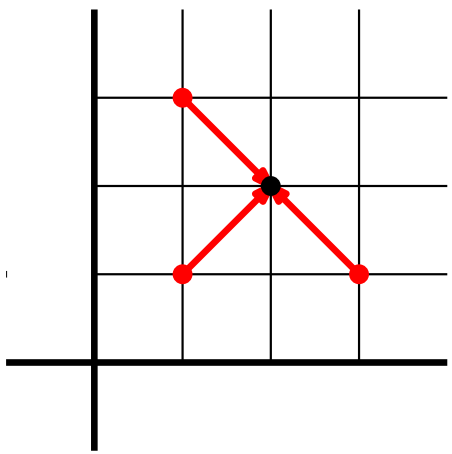


we define the *characteristic* function

$$S(x,y) := txy + \frac{ty}{x} + \frac{tx}{y}$$

A functional equation

$$Q(x,y;t) \quad = \quad 1 + S(x,y)Q(x,y;t)$$



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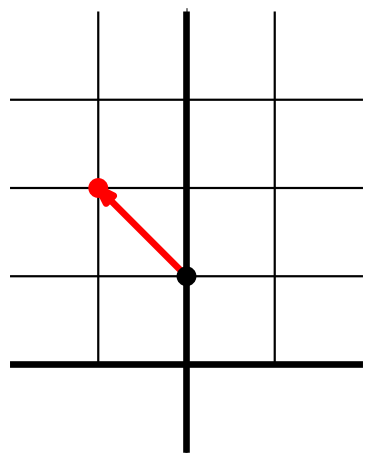

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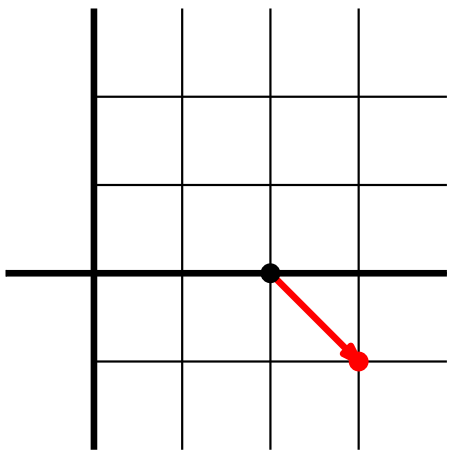

The Kernel Equation

Re-group terms and multiply by xy to give:

$$K(x,y)\cdot xyQ(x,y)=xy-ty^2Q(y,0)-tx^2Q(x,0) \qquad (\mathcal{K})$$

where $K(x,y) = 1 - S(x,y)$ is called the *kernel* of the walk.

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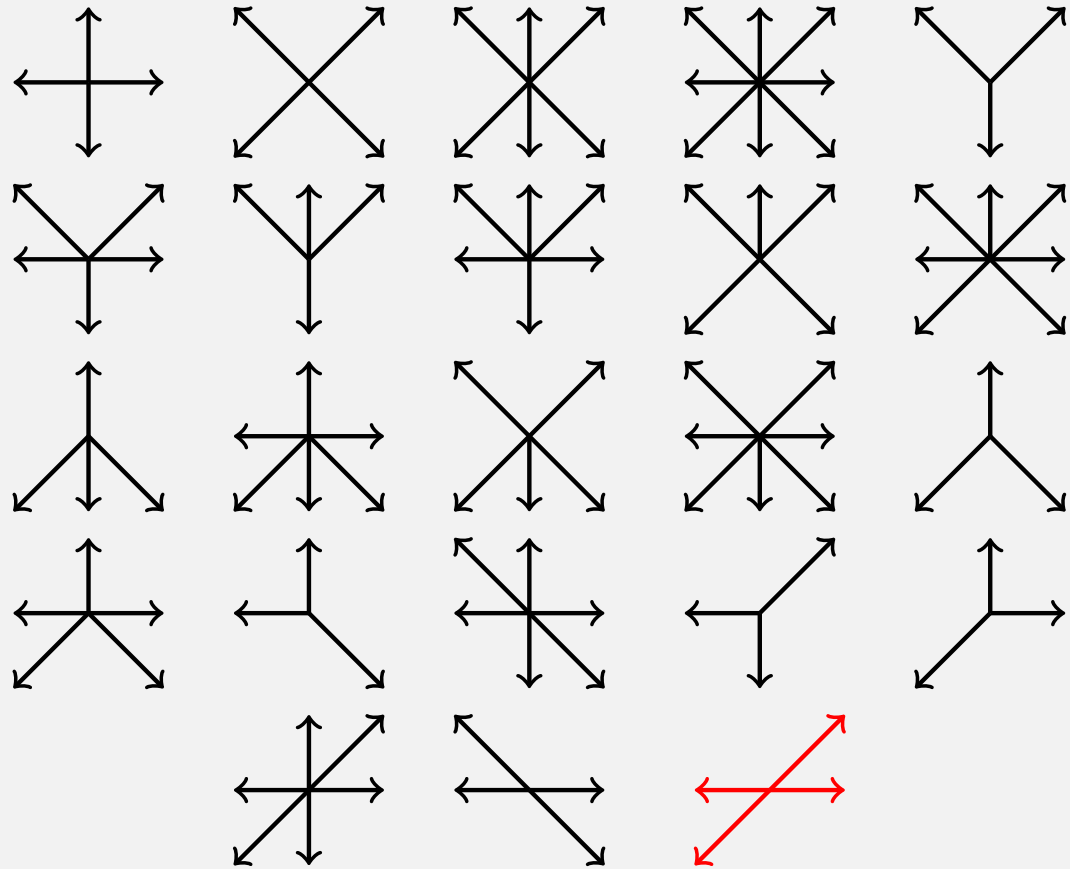
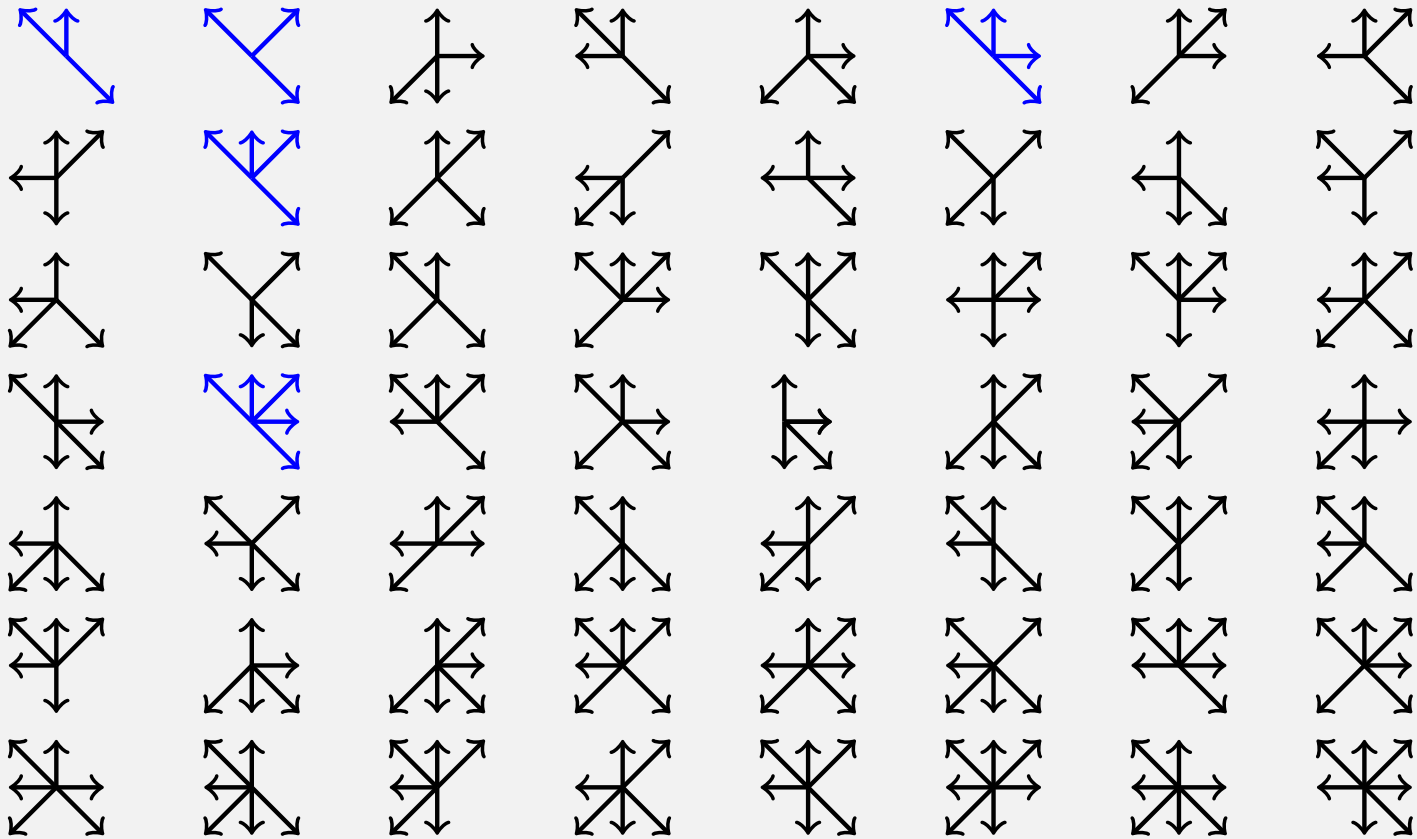
Related to a group G_S of transformations of the plane.

Finite $|G_{\mathcal{S}}|$ = D-finite GF



D-Finite by [Bousquet-Mélou & Mishna 2010]
and [Bostan & Kauers 2010]

Infinite $|G_{\mathcal{S}}|$ = non D-finite GF?



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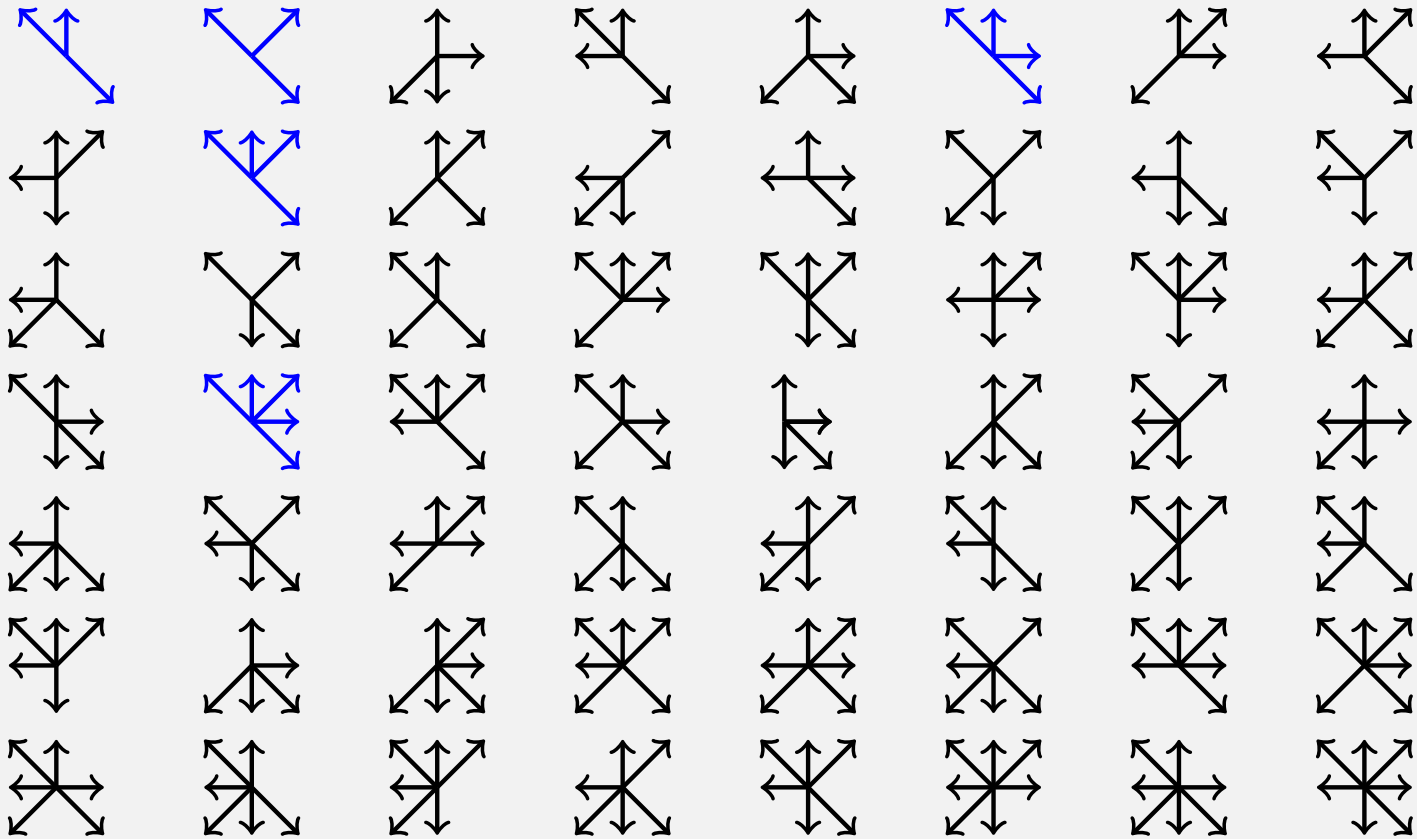

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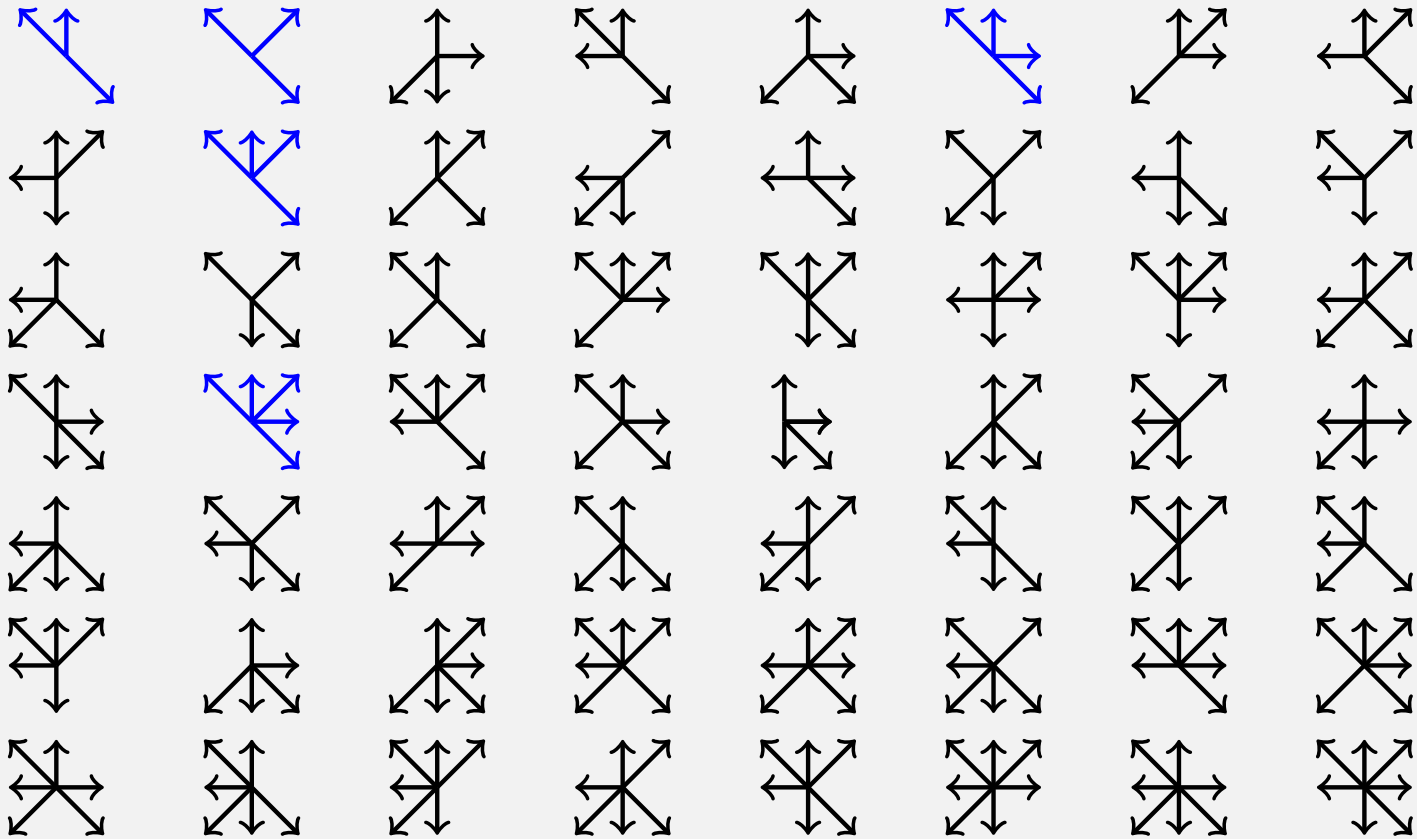

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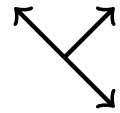
[M. & Mishna 2013]

The Iterated Kernel Method

An Explicit Formula

Recall, for the step set



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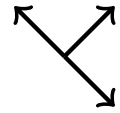

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Note that

$$Q(1,1) = \frac{1 - 2tQ(1,0)}{1 - 3t}.$$

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Let $Y_1(x;t)$ be the root of $K(x,y)$ with $Y_1 = xt + O(xt^2)$.
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Theorem (Mishna & Rechnitzer 09; M. & Mishna 13)

For the three symmetric singular models

$$Q(x,0) = \frac{1}{x^2 t} \sum_{n \geq 0} (-1)^n Y_n(x;t) Y_{n+1}(x;t).$$

Implications

Corollary (Fast Generation)

It is possible to calculate $q_0, q_1, \ldots, q_N$ *in* $\tilde{O}(N^3)$ *operations for each symmetric singular model.*

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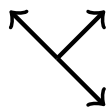
Corollary (Asymptotics)

The following asymptotic estimates hold:



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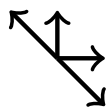
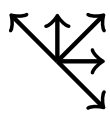


$$\kappa_B 4^n + O\left(\left(1+2\sqrt{2}\right)^n\right)$$



$$\kappa_C 5^n + O\left(\left(1+2\sqrt{3}\right)^n\right)$$

Extends results of [Fayolle & Raschel 2011].

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For each model $1/Y_n$ satisfies a linear recurrence of order 2 – solving gives an explicit form.

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Ex: For the three step model

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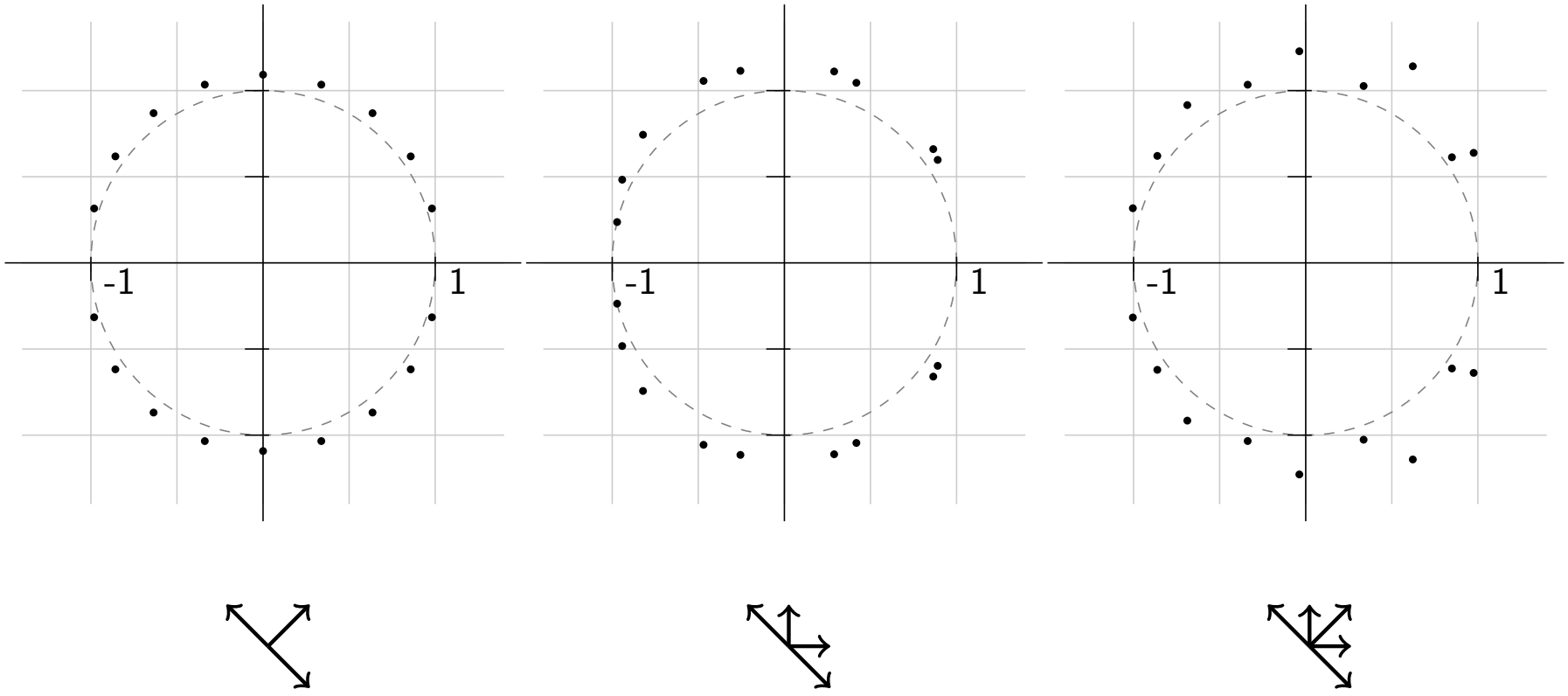
Proof Idea.

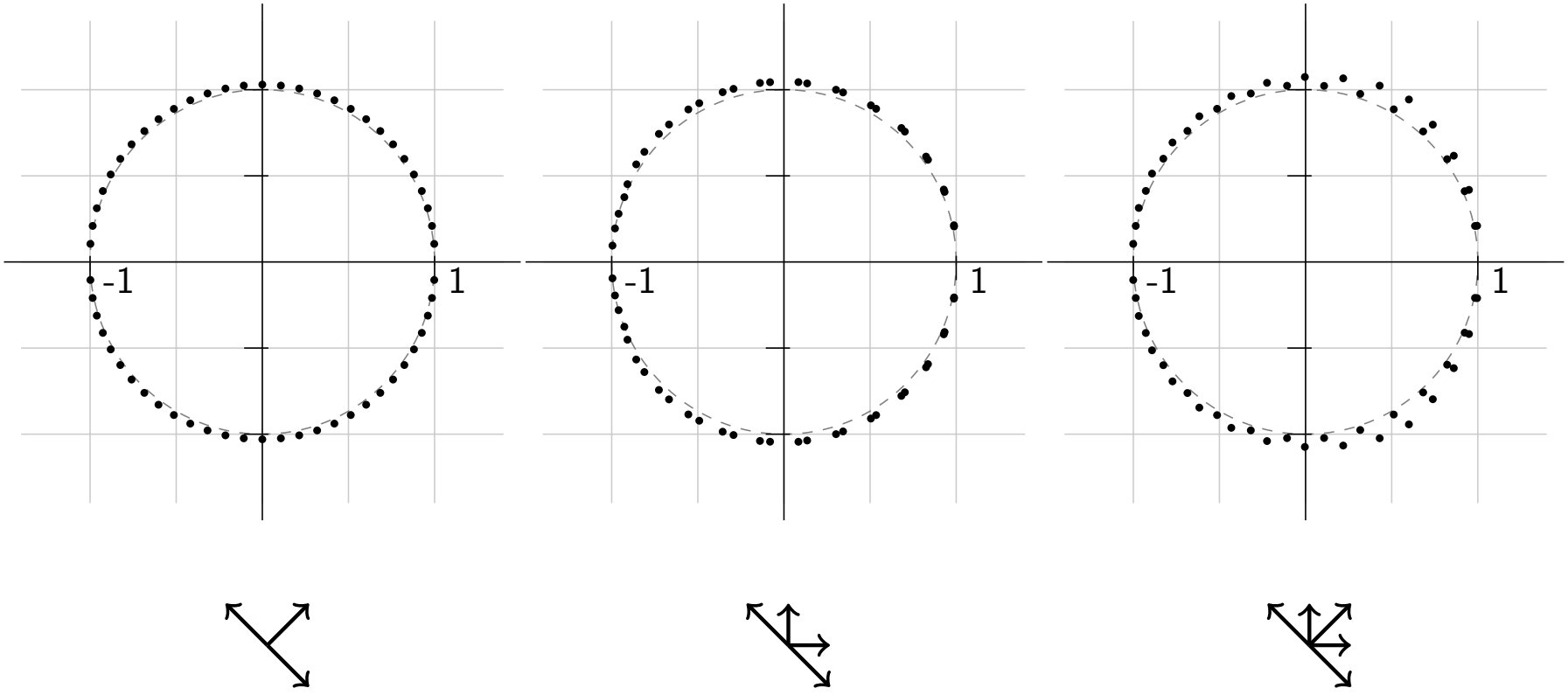
Each Y_n contributes a unique singularity. □

Singularities



$n = 10$

Singularities



$n = 30$

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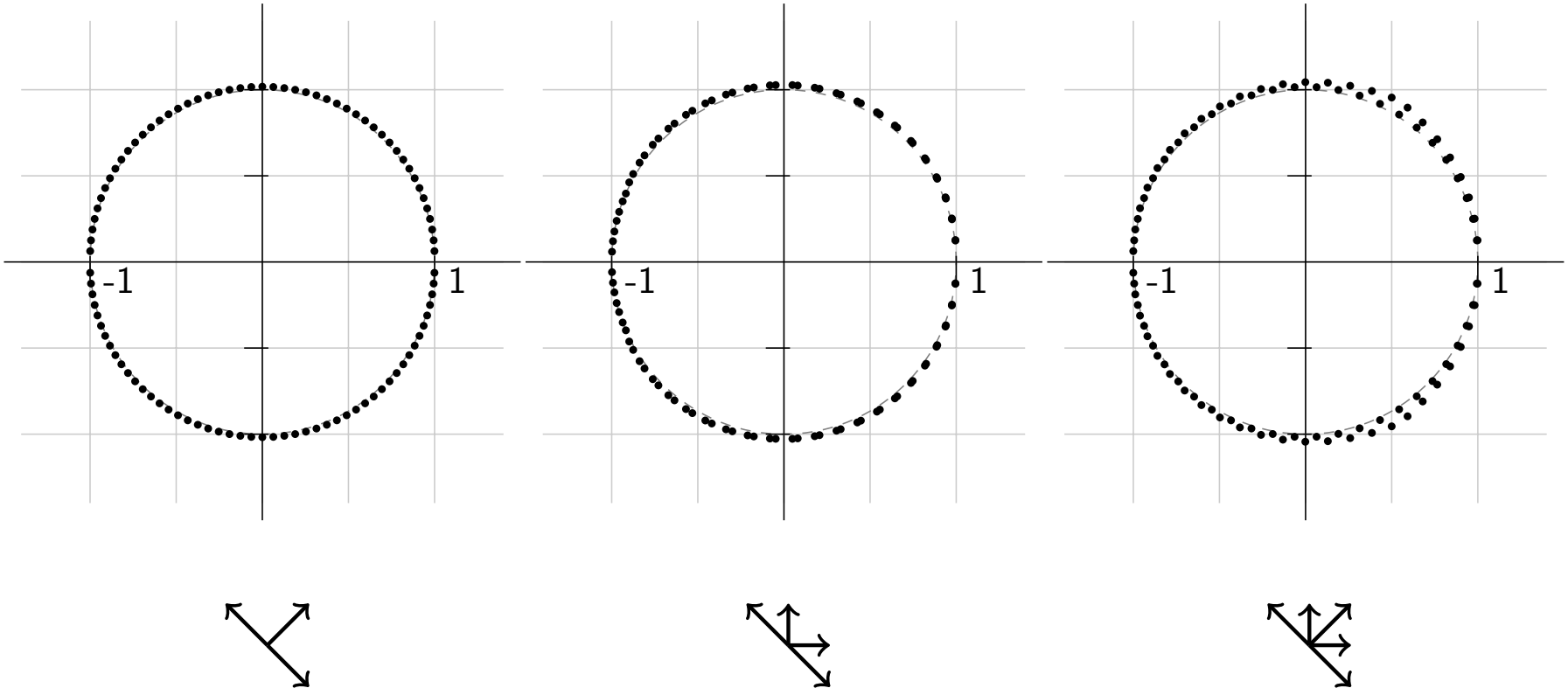

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Uniformization used by Kurkova, Fayolle, and Raschel gives another parametrization of Y_n.

Uniformization

For explicit functions x_3, x_4, τ, and s_0:

$$Y_n\left(\frac{q}{1+q^2}\right) = \frac{4x_3x_4}{x_4 - x_3} \cdot \frac{q^n\tau}{(q^n\tau - s_0)(q^n\tau - 1/s_0)}.$$

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Implies poles satisfy $q^n = s_0/\tau$ or $q^n = 1/(s_0\tau)$;

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Implies poles satisfy $q^n = s_0/\tau$ or $q^n = 1/(s_0\tau)$;

Gives simple proof singularities are distinct;

Uniformization

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Works even though the original method does not apply to singular walks.

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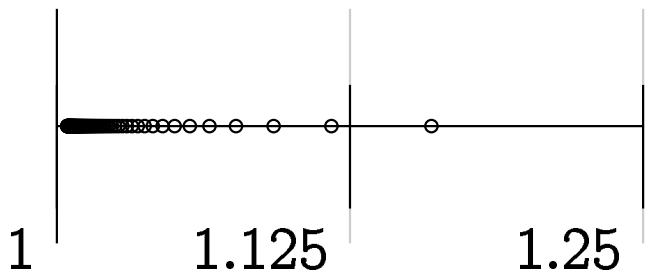
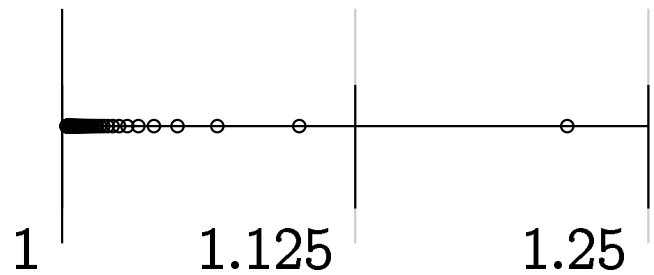
Luckily, there are an infinite number of singularities on the imaginary axis – in both cases.





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Cubic time algorithm for q_n.

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Extension to 3D walks

Problem: Getting terms to cancel

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Extension to 3D walks

Problem: Getting terms to cancel

Longer steps

Problem: No Uniformization

References

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