

ALEA 2013 - 18 Mars

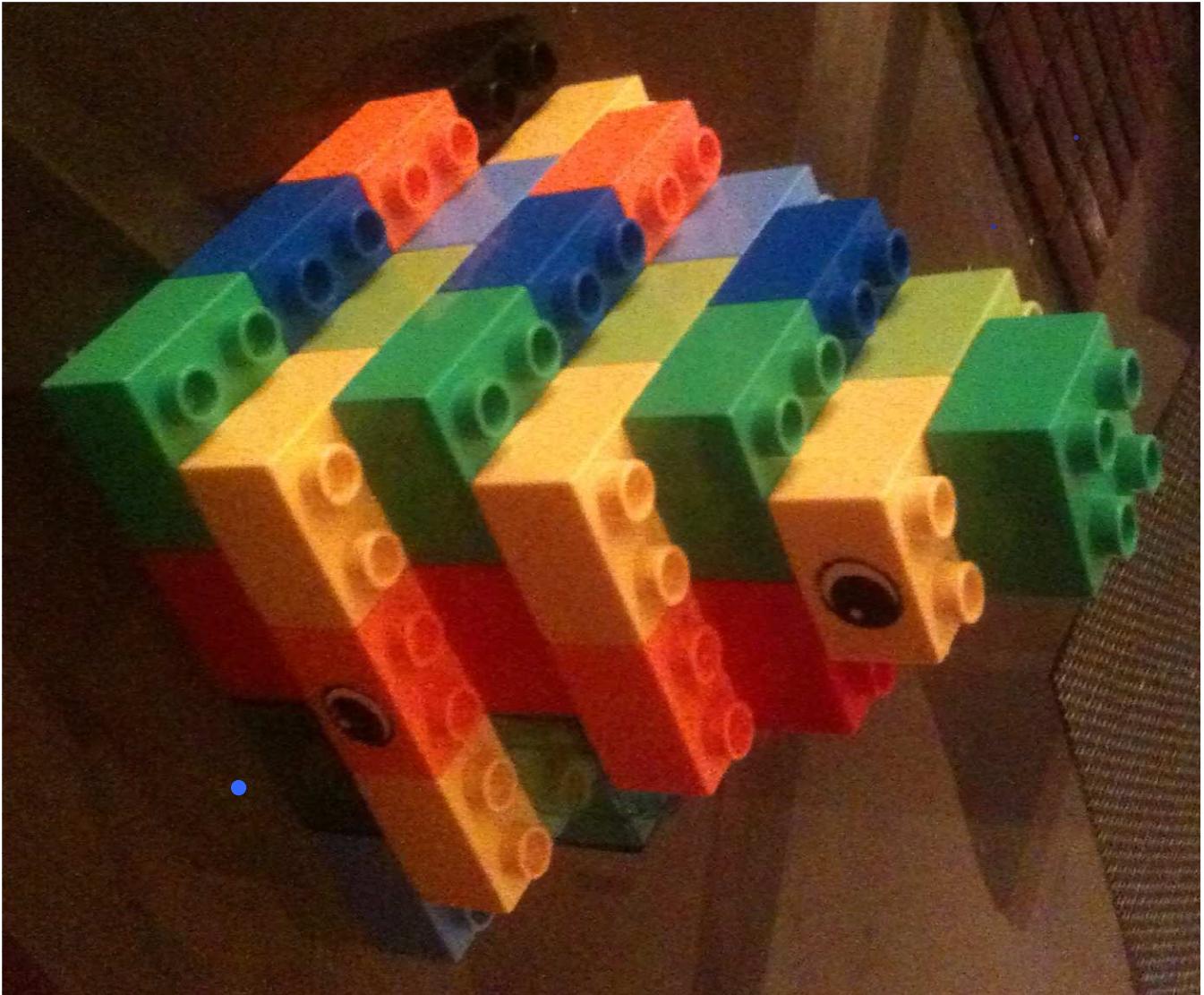
Titre de la note

15/02/2013

Pyramides et diamants

S. Cortee |
LIAFA CNRS

avec J. Boulier
et G. Chapuy

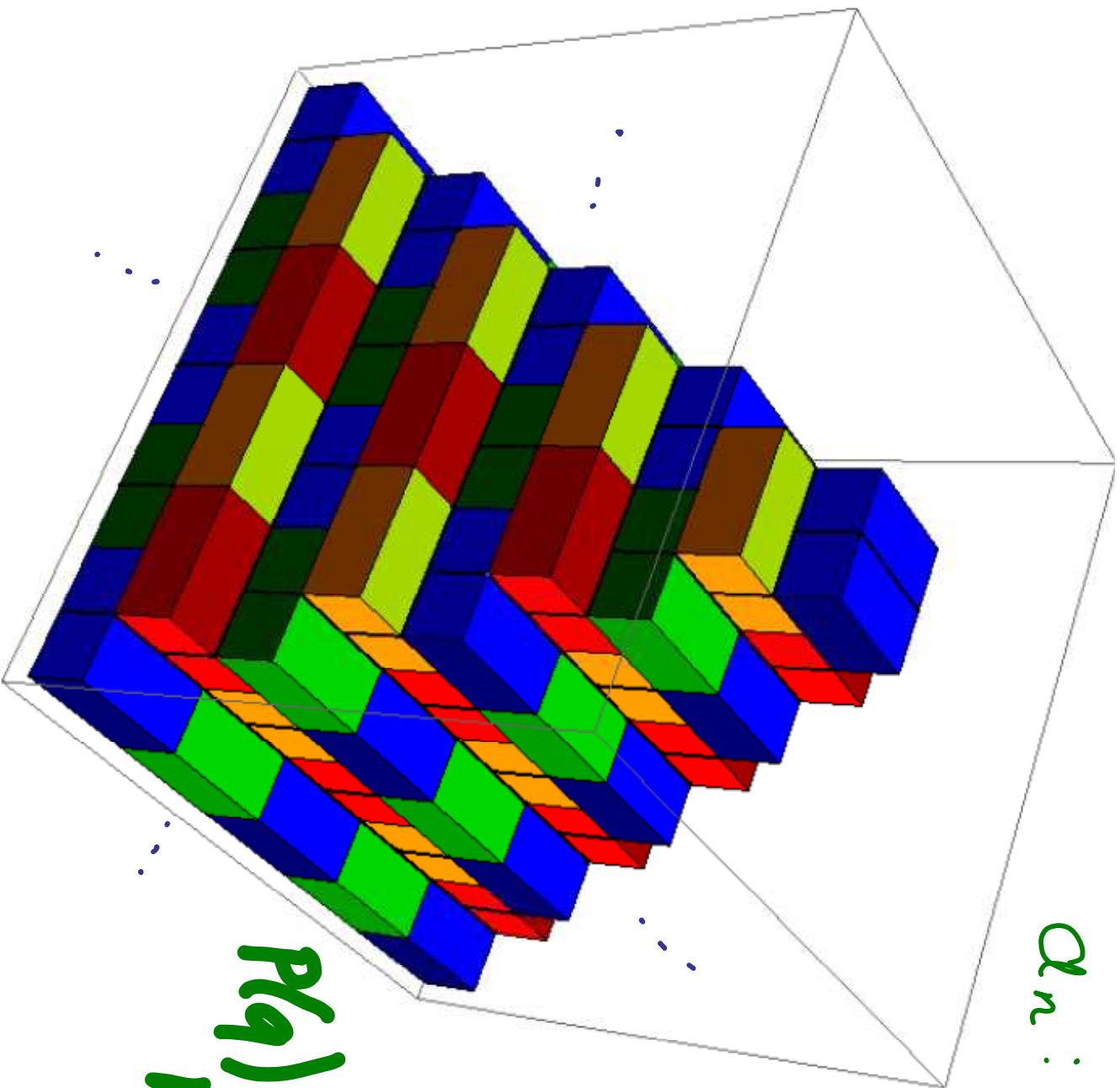


Pyramid partitions (Young OS)

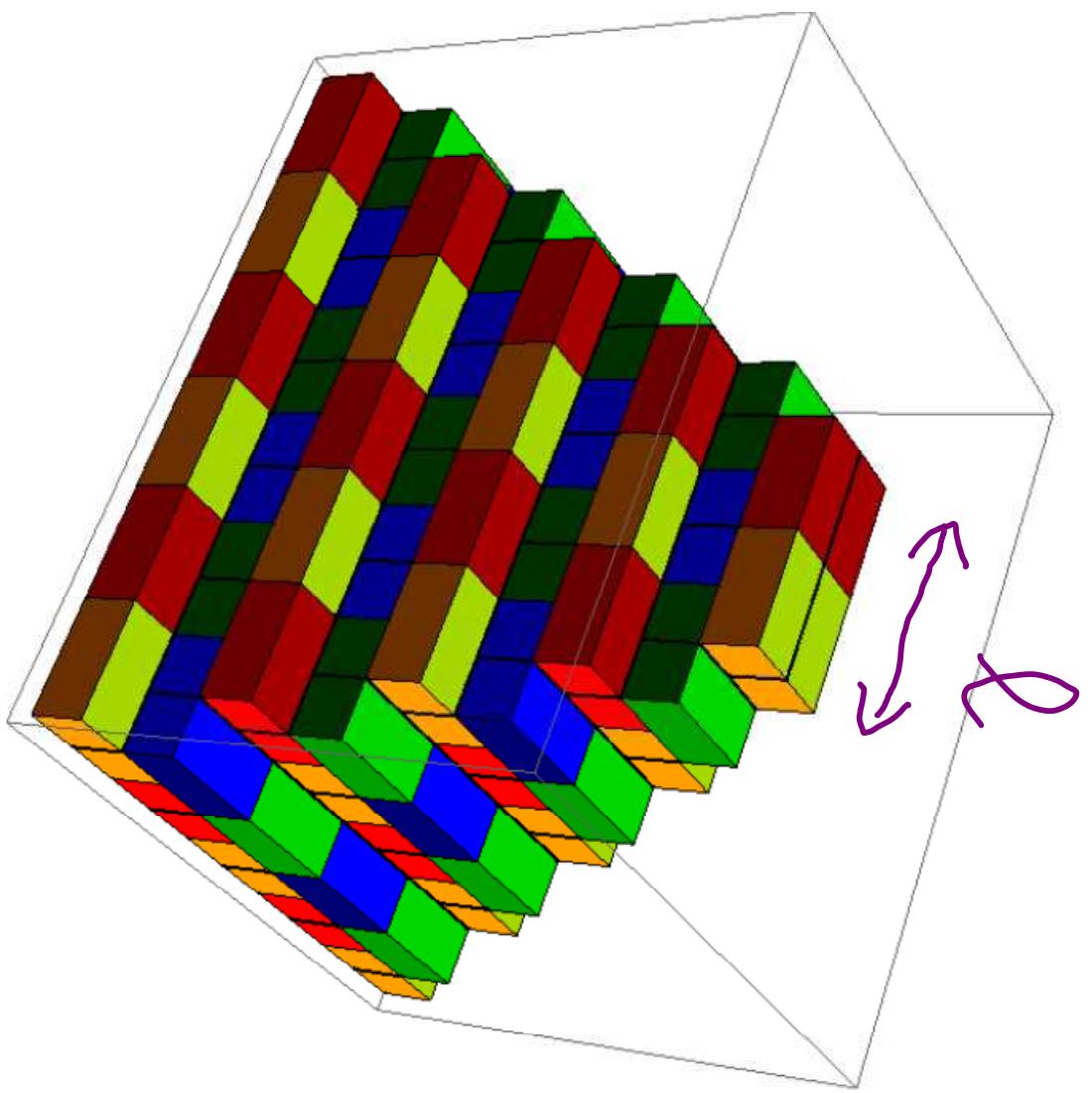
a_n : # de façons
de refirer
 n duplos

$$P(q) = \sum_{n \geq 0} a_n q^n$$

$$P(q) = \prod_{k \geq 1} \frac{(1 + q^{2k-1})^{2k}}{(1 - q^{2k})^{2k}}$$



ℓ -Pyramids



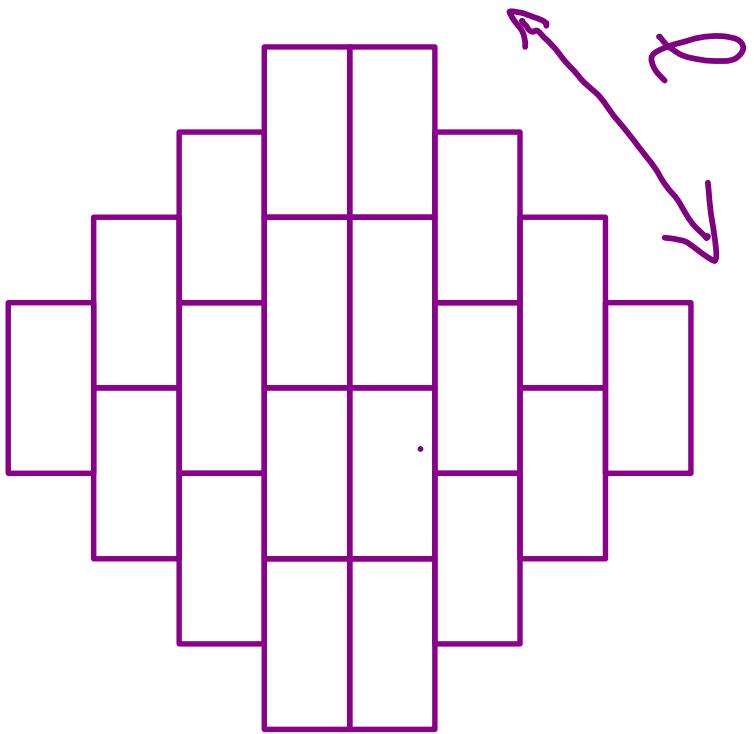
Thm (Young)

$$P_\ell(q) = \frac{P(q)}{A_{\ell-1}(q)}$$

$$A_\ell(q) = \prod_{k=0}^{\ell-1} (1 + q^{2|k|})^{\ell-k}$$

Aztec diamond (Eckies et al, 1992)

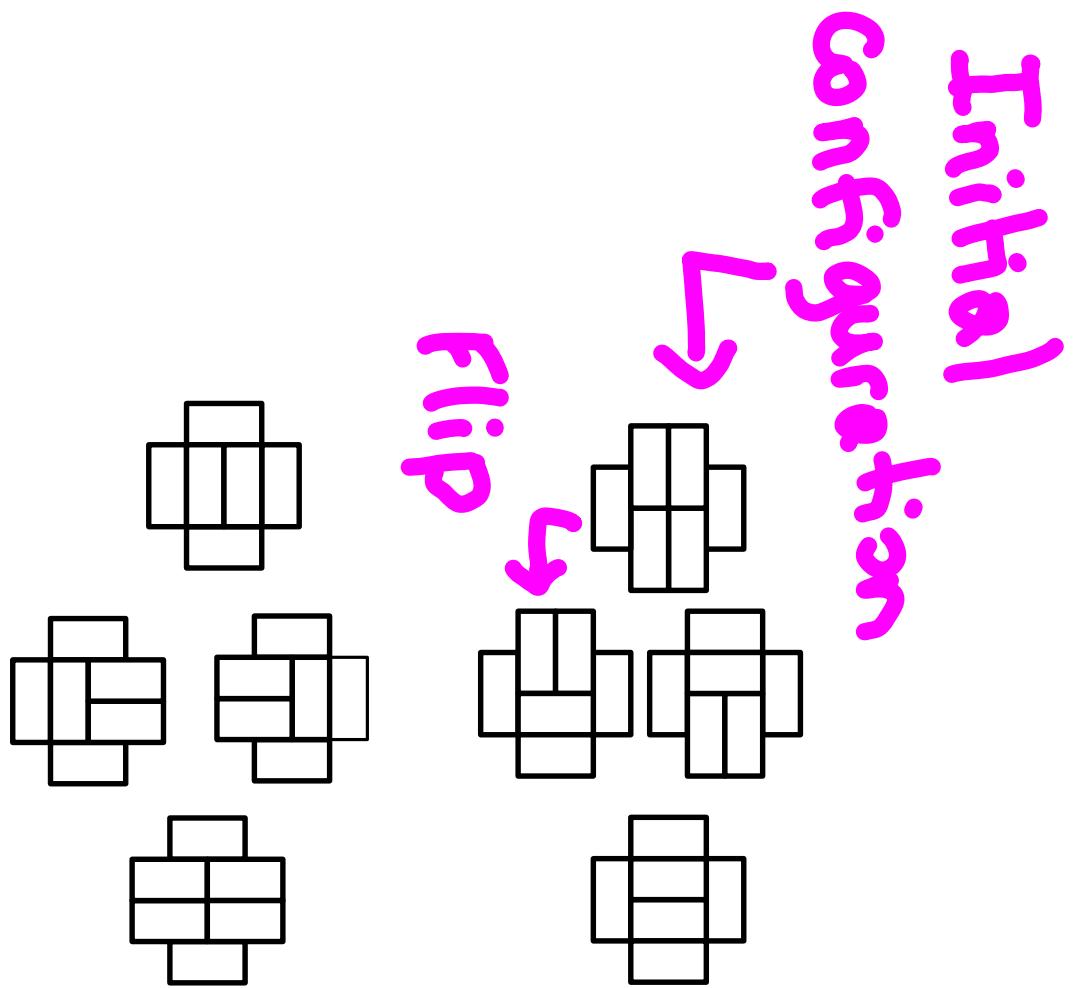
Initial configuration



FLIP



Example $\ell = 2$



8 tilings

Theorem (92)

$2^{(p+1)}$

2 tilings

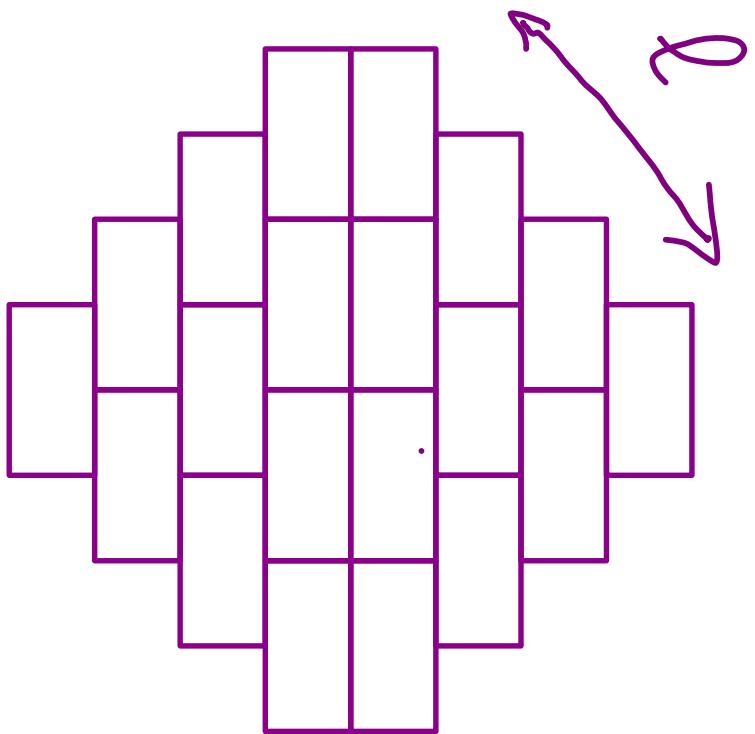
Aztec diamond (Eckies et al, 1992)

d_n : # of tilings after n flips

Flip generating func.

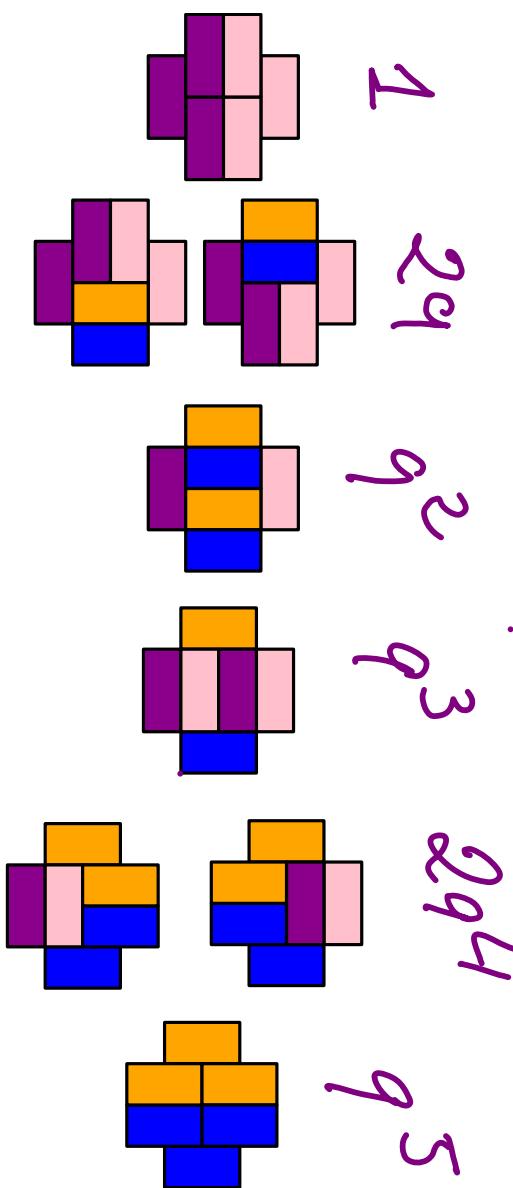
$$A_q(q) = \sum_{n=0}^{\infty} d_n q^n$$

$$A_q(q) = \prod_{j=0}^{t-1} (1 + q^{2j+1})^{d_j}$$

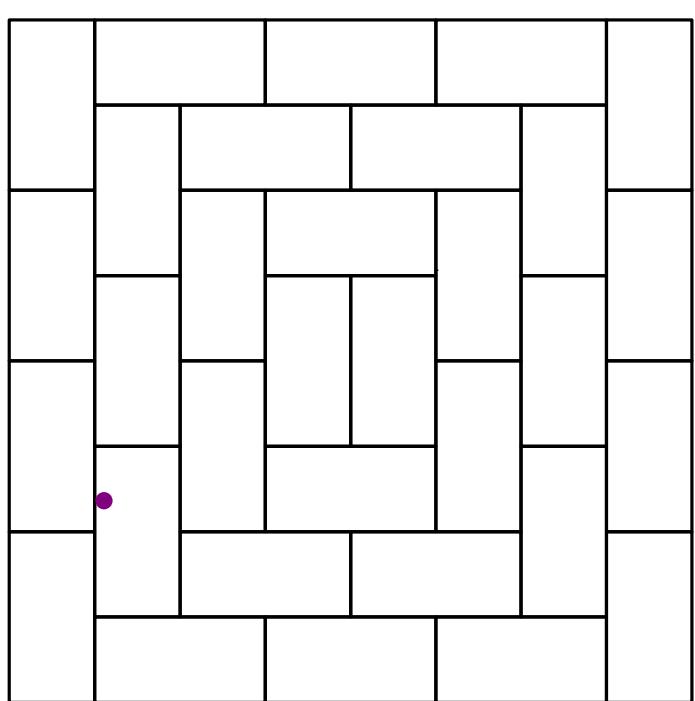


Example $l=2$

$$\begin{aligned}A_2(q) &= (1+q)^2 (1+q^3) \\&= 1 + 2q + q^2 + q^3 + 2q^4 + q^5\end{aligned}$$



Pyramid partitions



Initial tiling

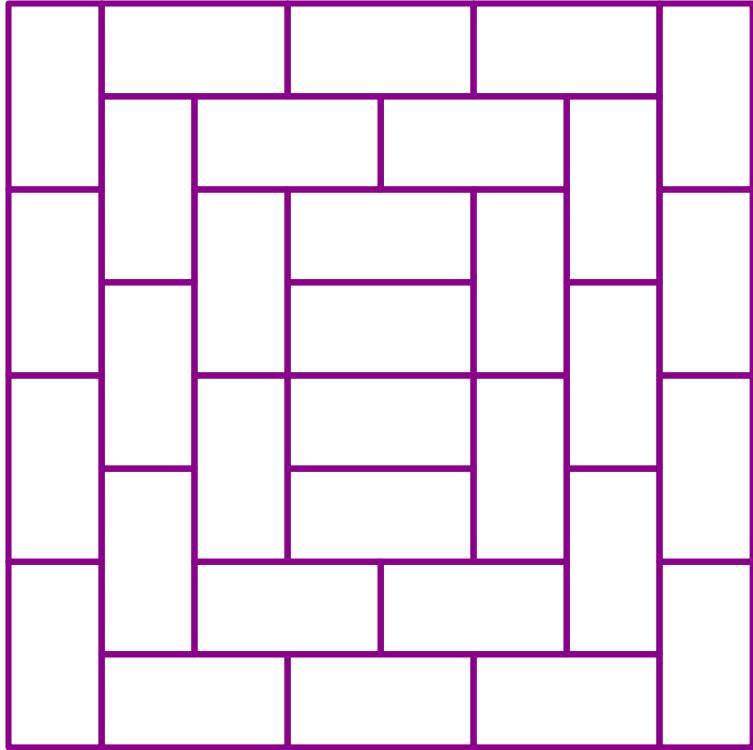
...

Plane

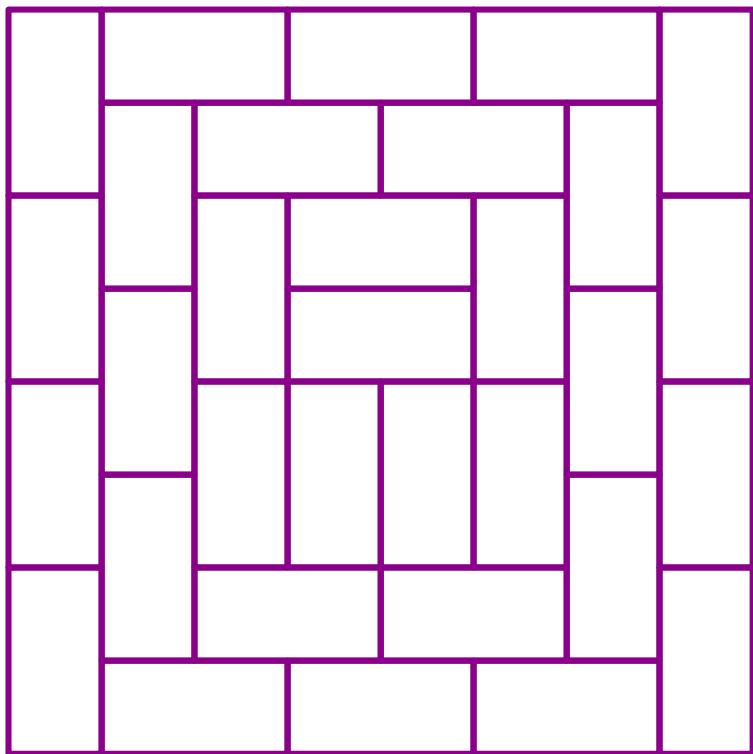
Domino T-tiling's
on the entire



Pyramid Partitioning → Tilings



1 flip



2 flips

$$a_n := \# \text{ tilings after } n \text{ flips}$$

$$= \frac{\prod_{k=1}^n (H_{2k-1})_{2k}}{\prod_{k=1}^n (1-q^{2k})_{2k}}$$

$$\sum_{n=0}^{\infty} a_n q^n$$

$$= \prod_{k=1}^{\infty} \frac{(H_{2k-1})_{2k}}{(1-q^{2k})_{2k}}$$

Goal

- Understand the global picture
- Are those families of living part of the same families?

Result (Bouffier, Chapuy, C.)

Given a word on length 2ℓ on the alphabet $\{+, -\}$

$$w = (w_1, \dots, w_{2\ell})$$

there exists a family of domino tilings

whose flip generating function is

$$f_w(q) = \prod_{\substack{i < j \\ w_i = +, w_j = -}} (1 + \epsilon_{ij} q^{j-i})^{\epsilon_{ij}}$$
$$\epsilon_{ij} = \begin{cases} 1 & j-i \text{ odd} \\ -1 & j-i \text{ even} \end{cases}$$

Remark

This is a hook formula

$$w = (+, -, -, +, +, -, -)$$

$$f_w(q) = \frac{(1+q^2)(1+q^3)(1+q^5)}{(1-q^2)^2(1-q^6)}$$

	1		+
2		-	
	2		+
5	2	1	+
6	3	2	-

$$w \mapsto \lambda(w)$$

$$f_w(q) = \prod_{\substack{x \in \lambda \\ h(x) \text{ odd}}} (1+q^{h(x)}) \prod_{\substack{x \in \lambda \\ h(x) \text{ even}}} \frac{1}{1-q^{h(x)}}$$

$$h(x) \text{ even}$$

Special cases

- $w = \underbrace{(+,-,\dots,+,-)}_{2\ell})$

Aztec diamond

- $w = (\dots, +, +, -, -, \dots)$

Pyramid partitions

- $w = (\dots, +, +, (+,-), -, -, \dots)$

ℓ -pyramid partitions

1	7
3	1
3	0
0	0

What are those things?

$$w = (+, +, +, +, +, -, -, -, +, +)$$

Path

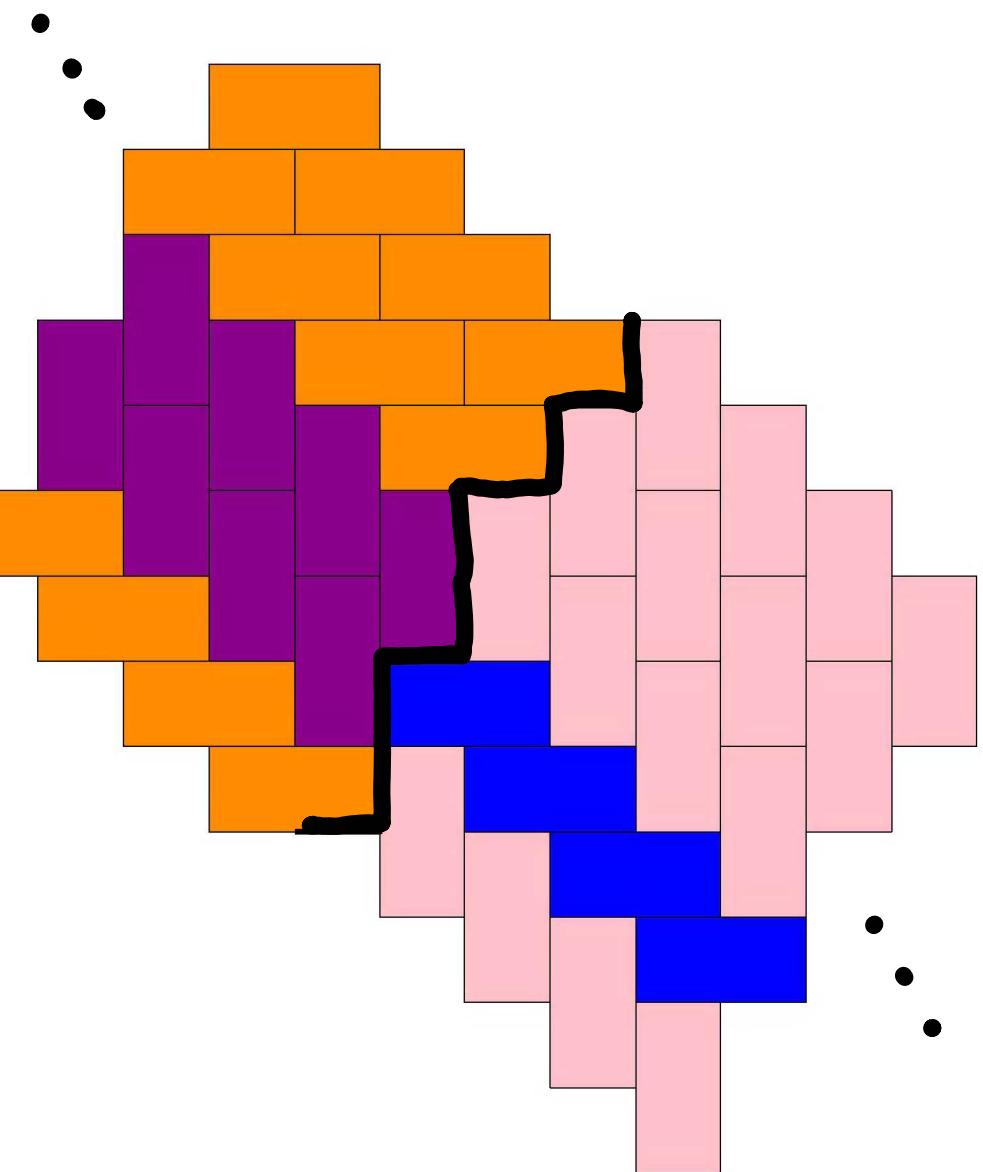
\sum

1
प्र०

- 2 even

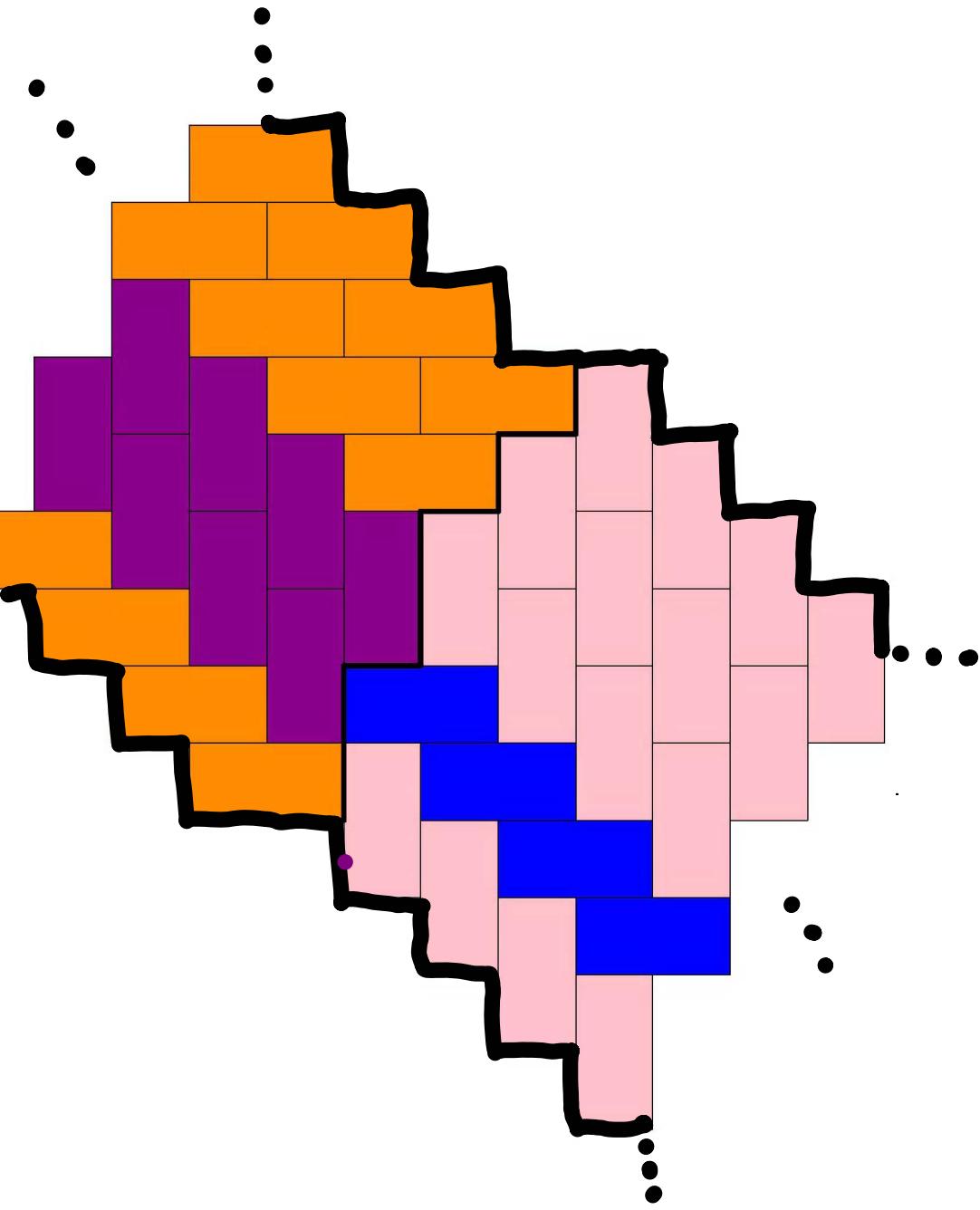
Σ
c.
||
|

— 1 —



$w = (+, +, +, +, +, -, -, -, -, +, +)$

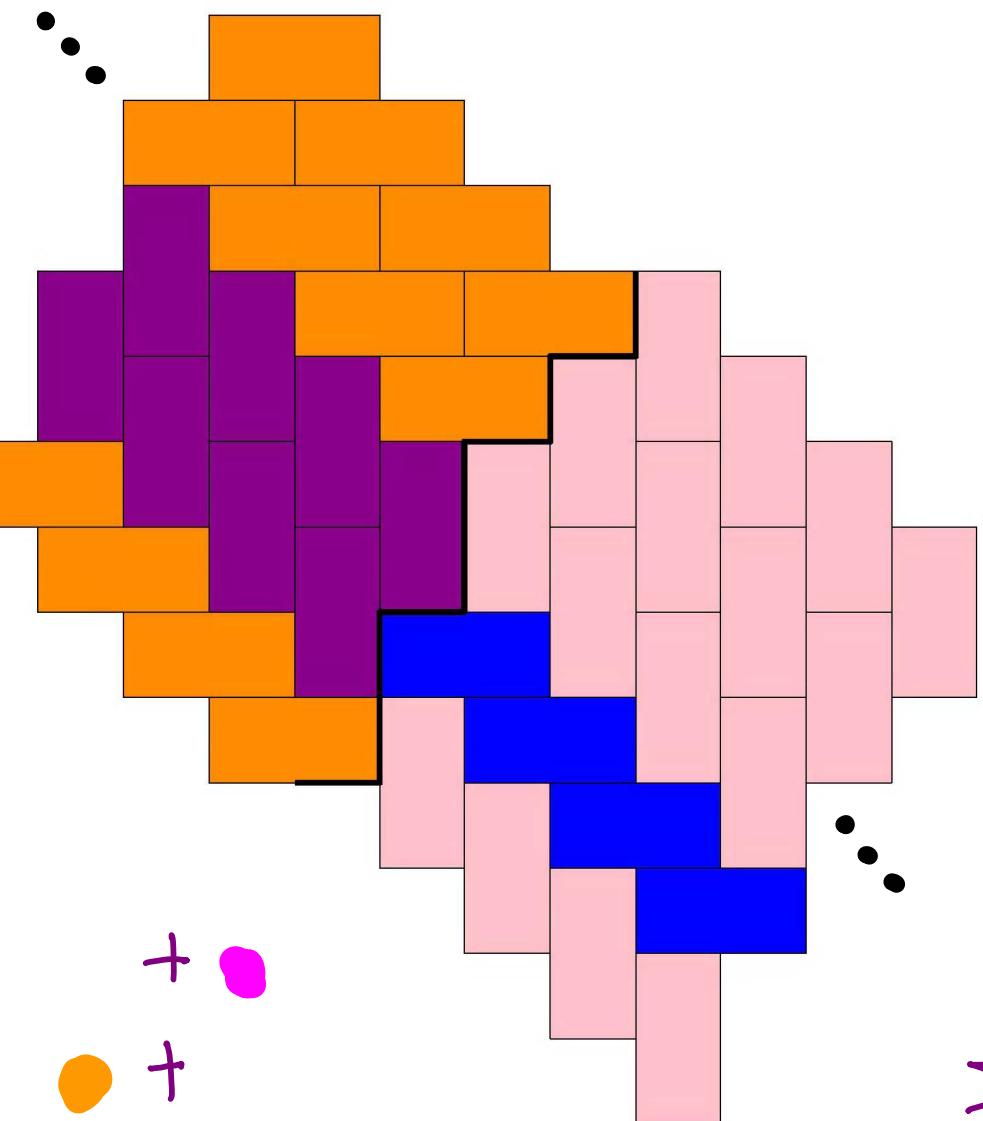
① Path



② Region

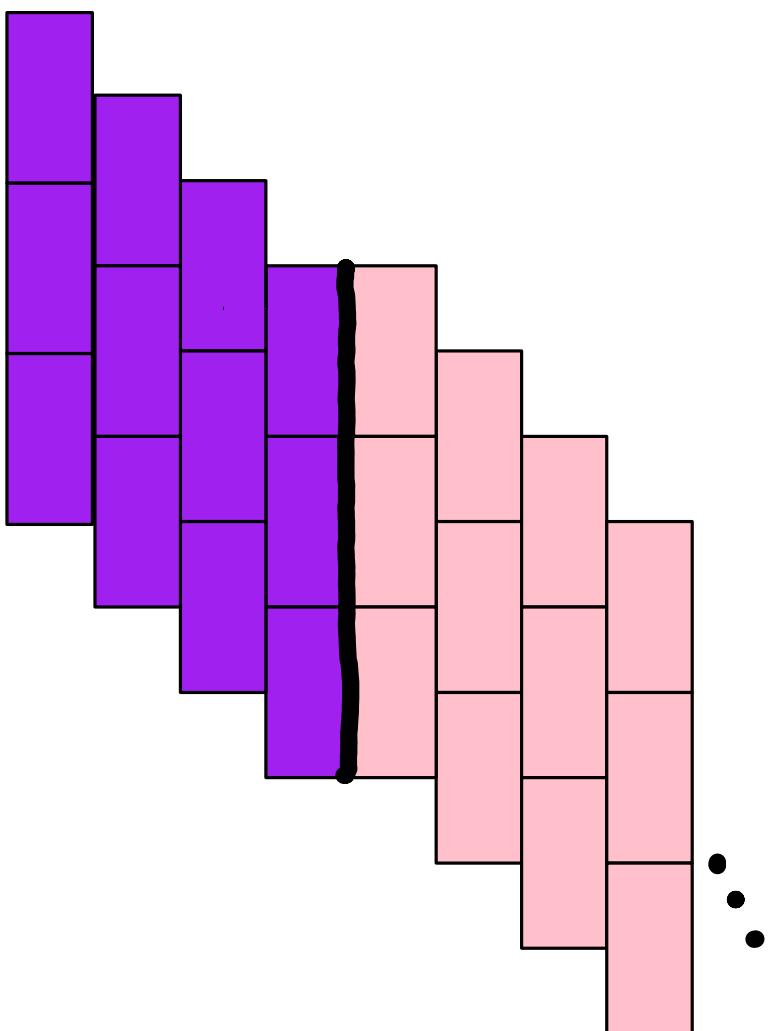
Minimal Living

above the path : pink and blue lilies
under the path : orange and purple lilies



A vertical column of hand-drawn symbols. It starts with an orange circle at the top, followed by a purple plus sign, a blue circle, another orange circle, another purple plus sign, another blue circle, a purple minus sign, another purple minus sign, another blue circle, a purple minus sign, another purple plus sign, another blue circle, and finally an orange circle at the bottom.

Example



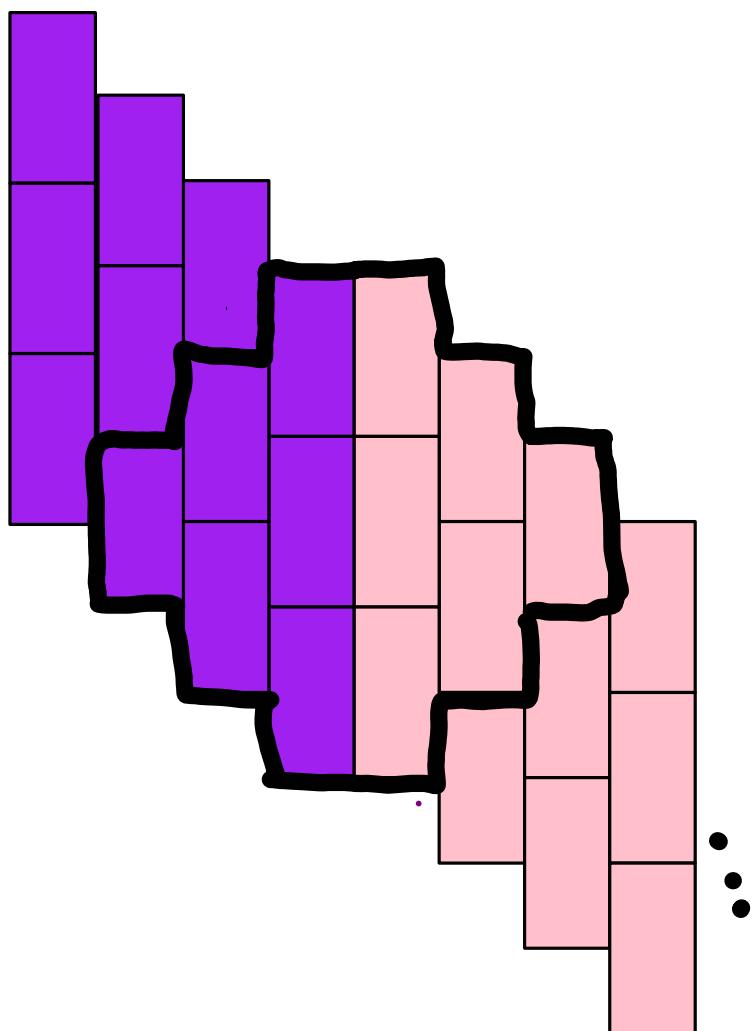
Aztec diamond of size 3

$$\text{Path} = (- - - - -)$$

$$w = (+, -, +, -, +, -)$$

Example

$$\omega = (+, -, +, -, +, -)$$

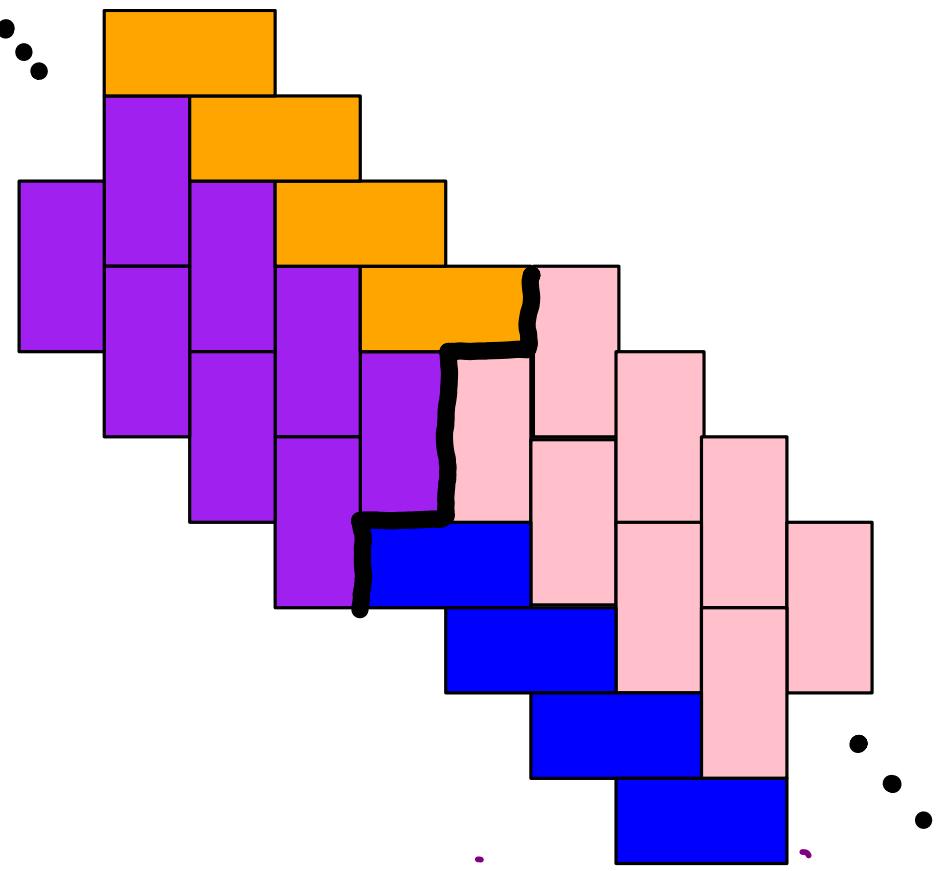


Aztec diamond of size 3
Path = (— — — — —)

Example 2

$$w = (+, +, +, -, -, -, -)$$

Semi Pyramidal Markings



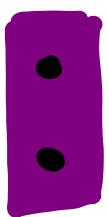
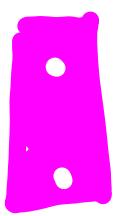
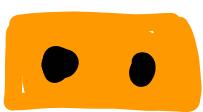
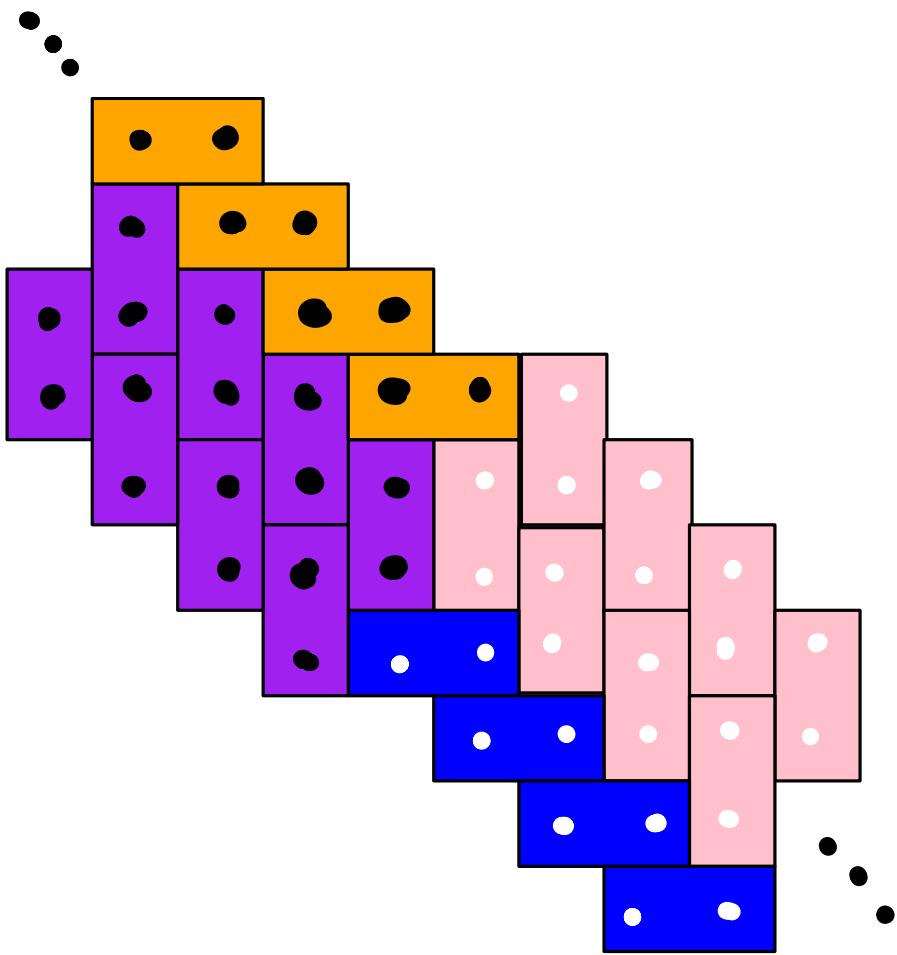
Proposition

There exists a bijection between tilings defined by \mathcal{W} of length n

- Sequences of integer partitions $\lambda^{(e)}, \dots, \lambda^{(2e)}$ with "some conditions"

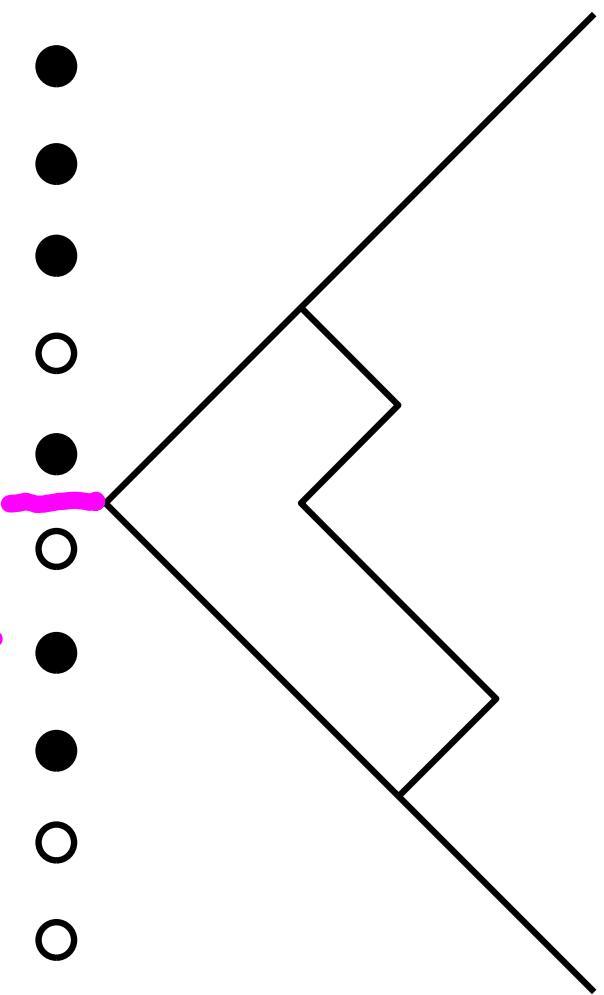
such that the min number of flips is $(|\lambda^{(e)}| + \dots + |\lambda^{(2e)}|)$.

From tilings to Particles



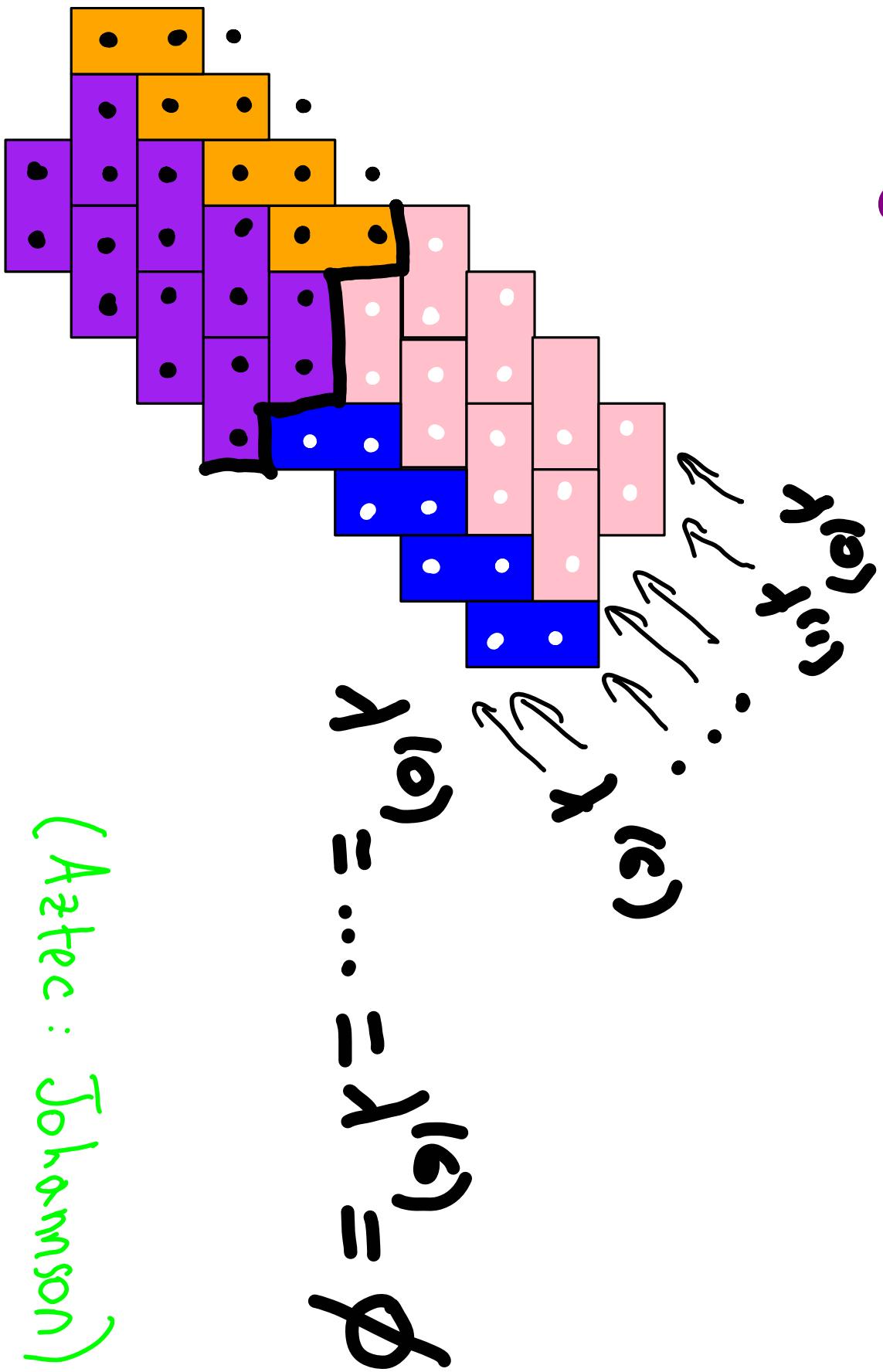
From particles to integer
partitions

$$\lambda = (3, 1)$$



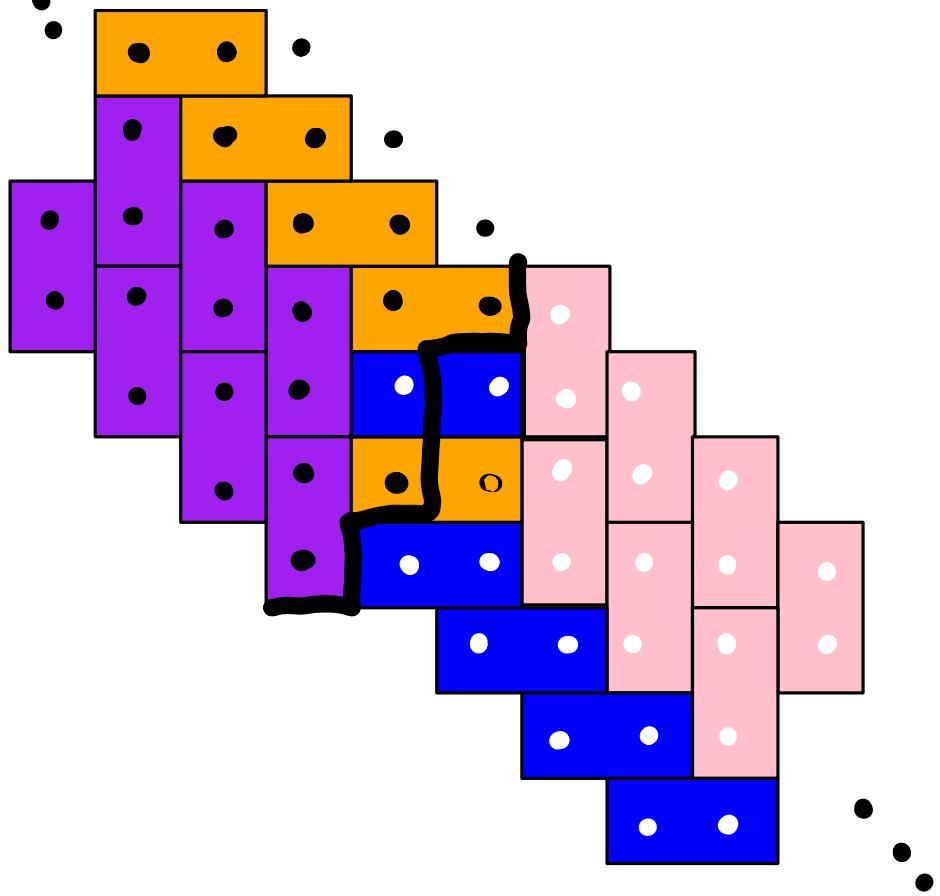
Young diagram

From tilings to sequences of integer partitions



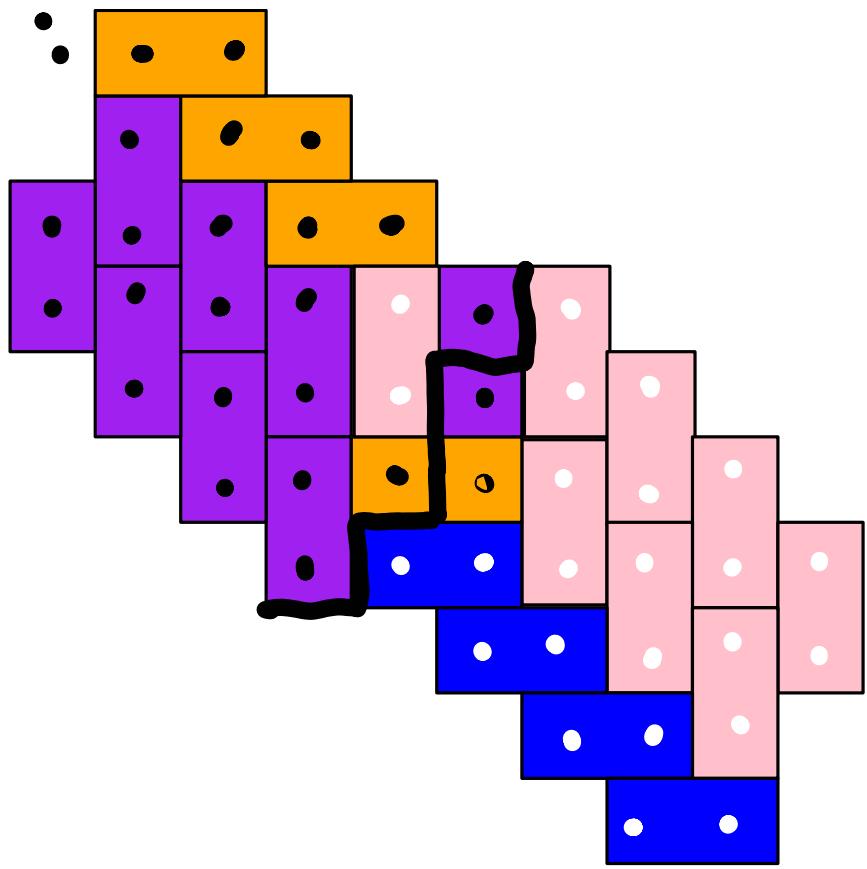
(Aztec : Johansson)

One
flip



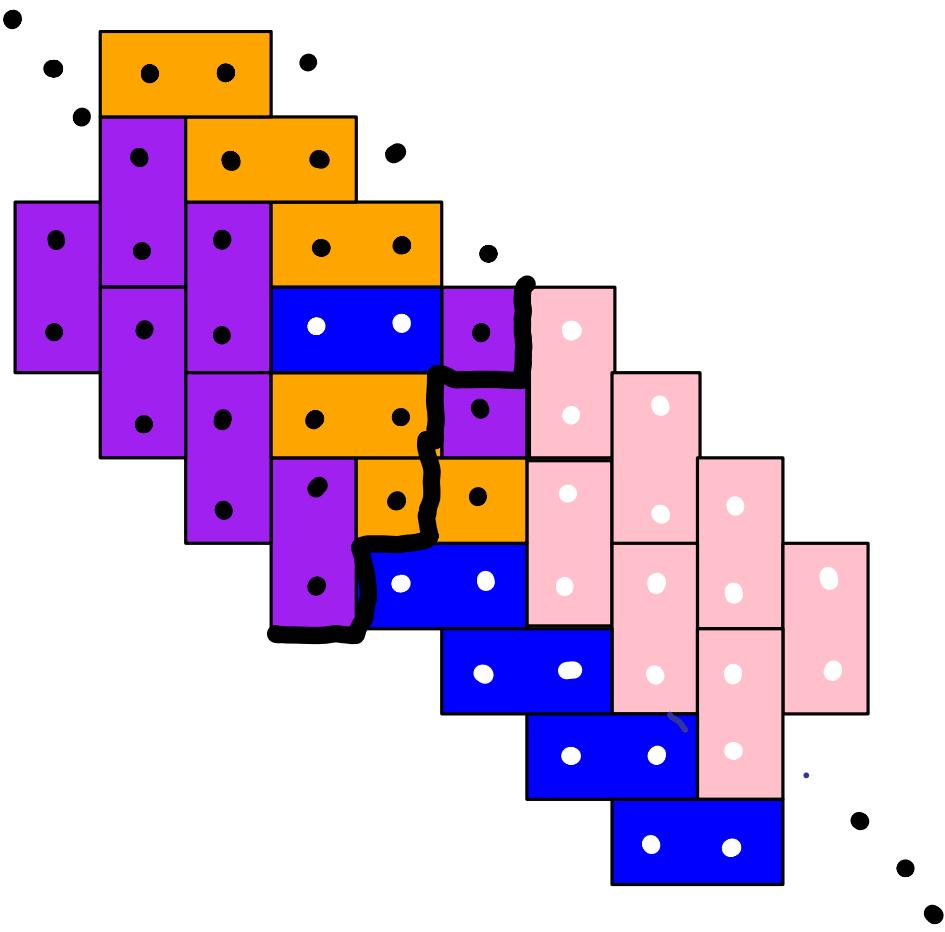
$$\begin{aligned}\lambda^{(0)} &= \lambda^{(1)} = \emptyset = \lambda^{(2)} \\ \lambda^{(3)} &= (\lambda) \\ \vdots & \\ \lambda^{(n)} &= \dots = \lambda^{(n)} = \emptyset\end{aligned}$$

Two
flips



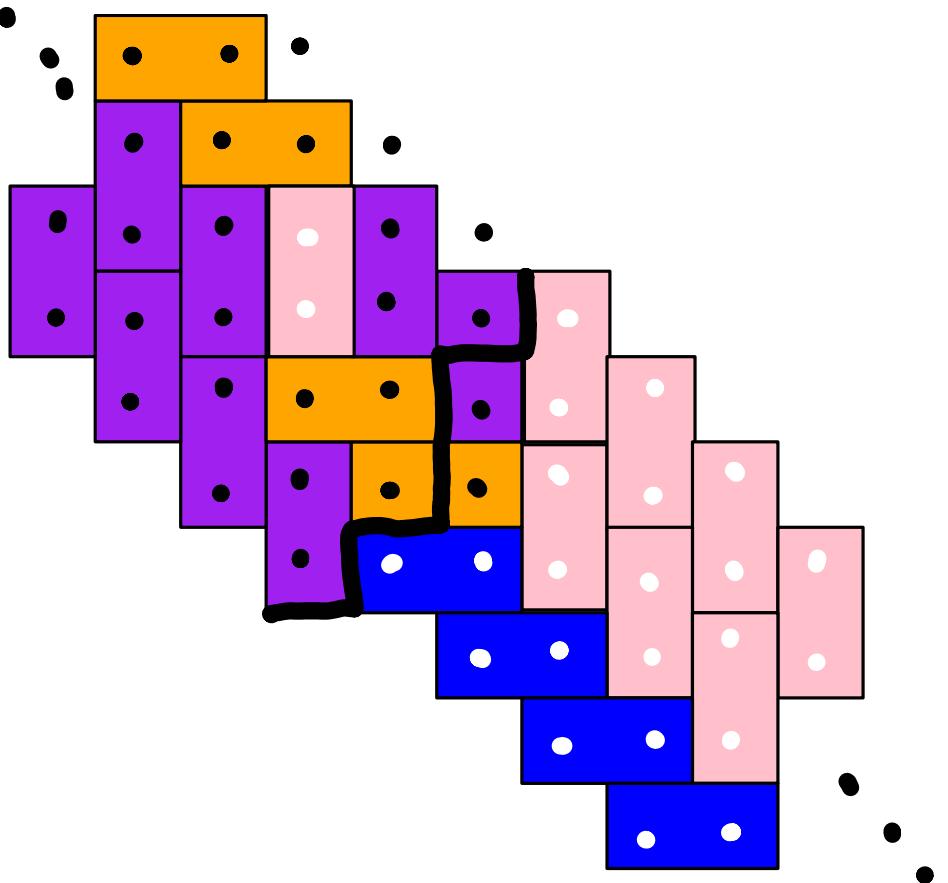
$$\lambda^{(2)} = \lambda^{(3)} = (\nu)$$

Three
flips



$$\chi^{(2)} = (1)$$
$$\chi^{(3)} = (1, 1)$$

Four
flips



After m flips

$\varnothing = \varnothing$ $\varnothing = \varnothing$

$\sum |X^{(i)}| = m$

$$X^{(1)} = (1, 1)$$

Proposition

• $\lambda^{(2i)}$ and $\lambda^{(2i-1)}$ differ by a vertical shift

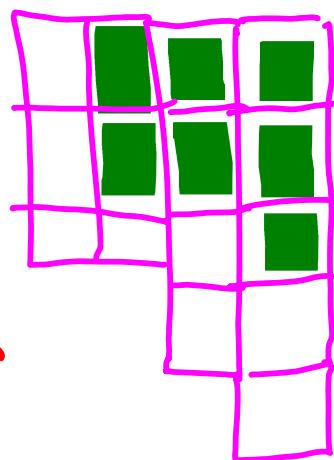
• $\lambda^{(2i+1)}$ and $\lambda^{(2i)}$ differ by a horiz. skip

Moreover

if $w_i = +$ then $\lambda^{(i)} \subseteq \lambda^{(i)}$
if $w_i = -$ then $\lambda^{(i)} \supseteq \lambda^{(i-1)}$

$$\lambda = (5, 4, 3, 3)$$

$$\alpha \subseteq \gamma$$



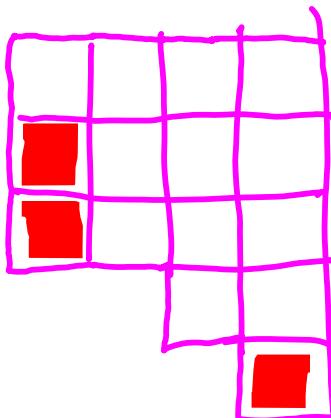
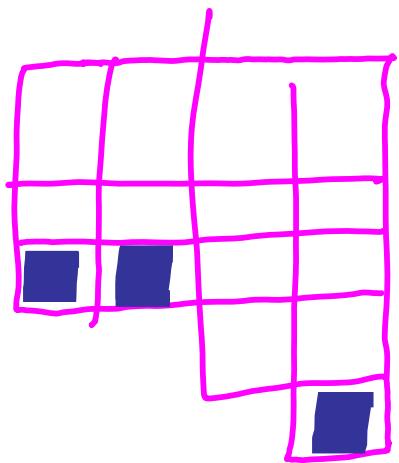
$$\mu = (4, 4, 3, 1)$$

λ / μ Horizontal ship



$$\nu = (4, 4, 2, 2)$$

λ / ν vertical ship



GOAL $w = (w_1, \dots, w_e)$

Flip generating function

generating function

$$F(q) = \prod_{i=1}^e (1 + \varepsilon_i q^{j-i})^{\varepsilon_i}$$

$$\varepsilon_{ij} = (-1)^{j-i-1}$$

$$w_i = x - m_j$$

-
- Vertex operations (O. R.)
 - Shuffling algorithms (E et al)
 - Growth diagrams (Fomin)
 - RSK type algorithms (Sagan, C, Savelief, Vuletic)

Why?

Super - Schur functions $S_\lambda(x,y)$

3

$\frac{1}{2} \quad \frac{1}{2}$

$\frac{1}{1} \quad \frac{1}{1}$

(Berele Remmel, Kratt)

x horz. strip

$\begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{matrix}$

x

Cauchy identity

$$\sum_{\lambda} S_\lambda(x,y) S_\lambda(w,z) = \prod_{i,j} \frac{(1+x_i z_j)(1+y_i w_j)}{(1-x_i w_j)(1-y_i z_j)}$$

Operators (Olojanek, Bondin, Okade/...)

$$\Gamma_+^*(z)|\lambda\rangle = \sum_{\mu < \lambda} z^{|\mu/\lambda|} |\mu\rangle$$

$$\Gamma_+(z)|\lambda\rangle = \sum_{\mu > \lambda} z^{|\mu/\lambda|} |\mu\rangle$$

$$\Gamma'_-(z) - \Gamma_-(z)$$

$$\begin{aligned}\mathcal{D}(q) \Gamma_+^*(z) &= \Gamma_+^*(zq) \mathcal{D}(q) \\ \mathcal{D}(q) \Gamma_+^*(z) &= -\Gamma_+(z/q) \mathcal{D}(q)\end{aligned}$$

Proposition (Bcc)

$$\omega = (\omega_1, \dots, \omega_e)$$

$$F(q) = \langle \phi | \prod_{i=1}^q D(q) \Gamma^{(1)}(q) D(q) \Gamma^{(2)}(q) | \phi \rangle$$

Commutation relations (MacDonald)

$$\Gamma^+_{\pm}(u) \Gamma^-_{\mp}(v) = \frac{1}{1-uv} \Gamma^-_{\pm}(v) \Gamma^+_{\mp}(u)$$

$$\Gamma^+_+(u) \Gamma^-_-(v) = (1+uv) \Gamma^-_-(v) \Gamma^+_+(u)$$

FLIP GF \square

Generalizations

• Change the region $\lambda^{(a)} = \gamma$ $\lambda^{(e)} = \mu$

• Follow the flips in each diagonal

• Mix vertical and horizontal strips \rightarrow plane partitions

$$\prod_{i < j} (1 + \epsilon_{ij} q_j^{-i}) \epsilon_i^j$$

$$\epsilon_{ij} = \begin{cases} -1 & \text{if } i < j \\ 1 & \text{if } i > j \end{cases}$$

$$w_i = +, w_j = -i$$

Other questions

- Random generation (Shuffling)
- Correlation functions (Borodin and Shlosman) Super Schur process
- Bijective proofs (Hilfmann Grassl , Krattenthaler)

• Limit
shape

