

A natural stochastic extension of the sandpile model on a graph

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- 1 Definition and differences with the Abelian sandpile model

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- 2 A characterisation of the recurrent states

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- 3 The lacking polynomial

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A vertex v is stable in the configuration η if $\eta_v \leq d^G(v)$. A configuration is stable if all its sites are stable.

Definition (2)

An unstable vertex topples. In the Abelian model (ASM), it sends one grain of sand to each of its neighbours. In the Stochastic model (SSM), it sends a grain to each neighbour independently with probability p for some $p \in (0, 1)$.

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The configuration η_{\max} is defined by $\eta_{\max}(v) = d^G(v)$ for all $v \in V$.

We define a Markov chain structure on the SSM. Let μ be a measure with support V . Let $(X_i, i \geq 1)$ be i.i.d. random variables with law μ . Start from any stable configuration η_0 . Given η_{i-1} for any $i \geq 1$, η_i is obtained as follows:

- Add a grain at position X_i to the configuration η_{i-1} . Let η'_i be the obtained configuration.
- Let η_i be the stabilisation of η'_i , that is $\eta_i = \text{RS}(\eta'_i)$ (in some cases no toppling is needed).

Recurrent configurations

The set of recurrent configurations for the Markov chain is the unique recurrent class containing η_{\max} . We denote this set by $\text{Sto}(G)$.

Recurrent configurations

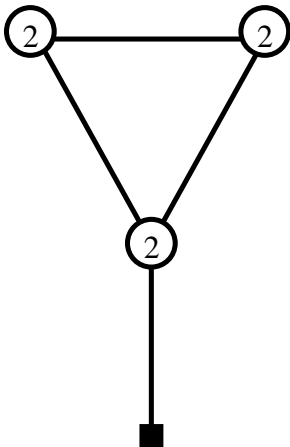
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Proposition

A configuration η is recurrent if and only if there exists a finite sequence of adding of grains and topplings such that η is reached from η^{\max} through this sequence.

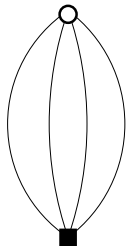
Differences between the ASM and the SSM

The set of recurrent configurations is not the same.



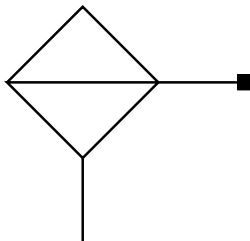
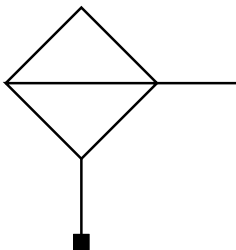
Differences between the ASM and the SSM (2)

The steady state measure on the SSM is not the uniform distribution.



Differences between the ASM and the SSM (3)

Changing the place of the sink can change the number of recurrent configurations for the SSM.



Characterisation of the recurrent configurations

An orientation of G is an orientation of each edge. For a configuration η , define the *lacking number* of η at v by $I_{\eta}^G(v) = d^G(v) - \eta_v$.

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Definition

We say that a configuration η on G is compatible with an orientation O of G , and write $\eta \in \text{comp}(O)$, if it is stable and

$$\forall v \in V, l_{\eta}^G(v) + 1 \leq \text{Out}_O^G(v).$$

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Theorem

We have

$$\text{Sto}(G) = \bigcup_O \text{comp}(O).$$

Characterisation of the recurrent configurations (2)

Theorem

A (stable) configuration η is recurrent (for the SSM) if and only if

$$\forall A \subseteq V, |E(G_A)| \leq \eta(A) - |A|.$$

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Theorem

A (stable) configuration η is recurrent for the ASM if and only if there exists an orientation O of G with no directed cycles such that $\eta \in \text{comp}(O)$.

The lacking polynomial - definition

For a configuration η , write $\ell(\eta) = \sum_v I_\eta^G(v)$ for the total number of grains lacking in η .

Definition

The *lacking polynomial* L_G of a graph G is the generating function of the recurrent configurations on G , with x conjugate to the number of lacking particles in the configuration:

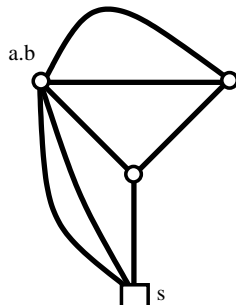
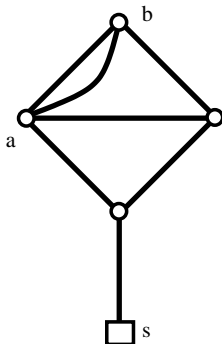
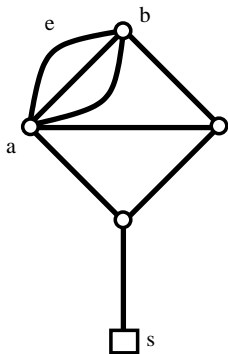
$$L_G(x) = \sum_{\eta \in \text{Sto}(G)} x^{\ell(\eta)}.$$

Definition

Let $G = (V \cup \{s\}, E)$ be a graph, and consider an edge $e = \{x, y\} \in E$, with $x, y \neq s$.

- 1 **Edge deletion.** The graph $G \setminus e$ is the graph G with e removed, i.e. $G \setminus e = (V \cup \{s\}, E \setminus \{e\})$.
- 2 **General edge contraction.** Define the graph $G.e$ as follows:
 - If e is simple, then $G.e$ is G with e contracted, i.e. $G.e = (V \cup \{x.y, s\} \setminus \{x, y\}, E \setminus \{e\})$, where edges adjacent in G to either x or y are now connected to $x.y$ instead.
 - If e has multiplicity $k \geq 2$, contract one of these edges as above, and replace the other $k - 1$ edges with $k - 1$ edges $\{x.y, s\}$.

Example



The recurrence relation

Theorem

Let e be an edge of E which is neither a bridge (i.e. removing e doesn't disconnect the graph), nor connected to the sink. Then

$$L_G(x) = xL_{G \setminus e}(x) + L_{G.e}(x).$$

Proposition

- ① If $V = \{u\}$ and there are k edges between u and s , then

$$L_G(x) = \sum_{i=0}^{k-1} x^i.$$

- ② If T is a tree attached to the rest of G at some vertex v , then $L_G = L_{G \setminus T}$.

- ③ If we can write G as the union of connected graphs $G_i = (V_i \cup \{s\}, E_i)$, $i = 1, \dots, k$, so that the V_i are mutually disjoint, then

$$L_G(x) = \prod_{i=1}^k L_{G_i}(x).$$

- Study the invariant distribution: seems very complicated.
- Generalise to directed graphs.
- Study the lacking polynomial in more detail: link with the Tutte polynomial, what does it count?

Thank you for your attention