

# The probability of planarity of a random graph near the critical point

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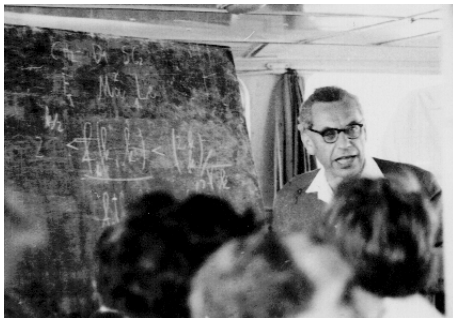
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# The material of this talk

- 1.— **Planarity on the critical window for random graphs**
- 2.— **Our result. The strategy**
- 3.— **Cubic planar multigraphs**
- 4.— **Other applications**

# Planarity on the critical window for random graphs



## The model $G(n, M)$

There are  $2^{\binom{n}{2}}$  labelled graphs with  $n$  vertices.

A random graph  $G(n, M)$  is the probability space with properties:

- ▶ **Sample space:** set of labelled graphs with  $n$  vertices and  $M = M(n)$  edges.
- ▶ **Probability:** Uniform probability  $\left(\binom{n}{2}^{-1}\right)^M$

### Properties:

- ♥ Fixed number of edges ✓
- ♣ The probability that a fixed edge belongs to the random graph is  $p = \binom{n}{2}^{-1} M$ . ✓
- ♠ There is not independence.

# The Erdős-Rényi phase transition

Random graphs in  $G(n, M)$  present a dichotomy for  $M = \frac{n}{2}$ :

1. – **(Subcritical)**  $M = cn, c < \frac{1}{2}$ : a.a.s. all connected components have size  $O(\log n)$ , and are either trees or unicyclic graphs.
2. – **(Critical)**  $M = \frac{n}{2} + Cn^{2/3}$ : a.a.s. the largest connected component has size of order  $n^{2/3}$
3. – **(Supercritical)**  $M = cn, c > \frac{1}{2}$ : a.a.s. there is a **unique** component of size of order  $n$ .

Double jump in the creation of the *giant component*.

# The problem; what was known

## ON THE EVOLUTION OF RANDOM GRAPHS

by

P. ERDŐS and A. RÉNYI

*Dedicated to Professor P. Turán at his 50th birthday.*

We can show that for  $N(n) = \frac{n}{2} + \lambda \sqrt{n}$  with any real  $\lambda$  the probability of  $\Gamma_{n, N(n)}$  not being planar has a positive lower limit, but we cannot calculate its value. It may even be 1, though this seems unlikely.

### **PROBLEM:** Compute

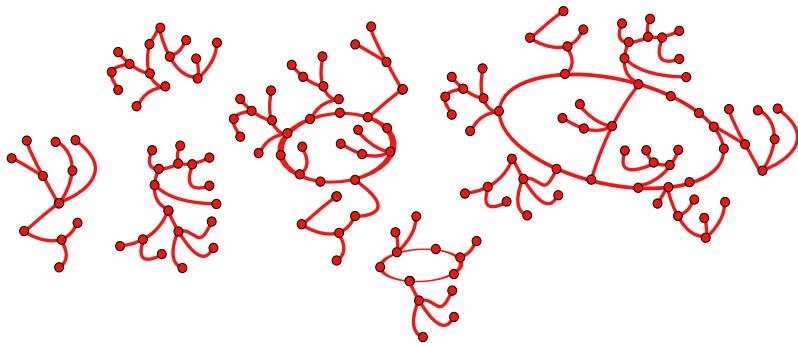
$$p(\lambda) = \lim_{n \rightarrow \infty} \Pr \left\{ G \left( n, \frac{n}{2} (1 + \lambda n^{-1/3}) \right) \text{ is planar} \right\}$$

What was known:

- ▶ Janson, Łuczak, Knuth, Pittel (94):  $0,9870 < p(0) < 0,9997$
- ▶ Łuczak, Pittel, Wierman (93):  $0 < p(\lambda) < 1$

**Our contribution:** the whole description of  $p(\lambda)$

# Our result. The strategy



## The main theorem

**Theorem (Noy, Ravelomanana, R.)** Let  $g_r(2r)!$  be the number of cubic planar weighted multigraphs with  $2r$  vertices. Write

$$A(y, \lambda) = \frac{e^{-\lambda^{3/6}}}{3^{(y+1)/3}} \sum_{k \geq 0} \frac{\left(\frac{1}{2} 3^{2/3} \lambda\right)^k}{k! \Gamma((y+1-2k)/3)}.$$

Then the limiting probability that the random graph  $G\left(n, \frac{n}{2}(1 + \lambda n^{-1/3})\right)$  is planar is

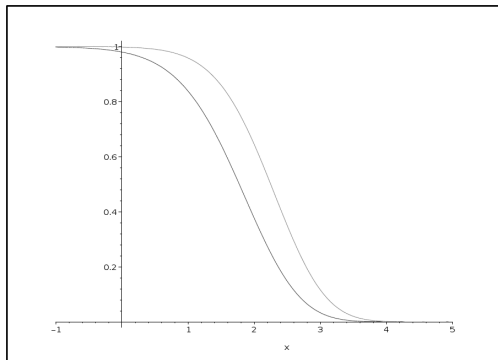
$$p(\lambda) = \sum_{r \geq 0} \sqrt{2\pi} g_r A\left(3r + \frac{1}{2}, \lambda\right).$$

In particular, the limiting probability that  $G\left(n, \frac{n}{2}\right)$  is planar is

$$p(0) = \sum_{r \geq 0} \sqrt{\frac{2}{3}} \left(\frac{4}{3}\right)^r g_r \frac{r!}{(2r)!} \approx 0,99780.$$

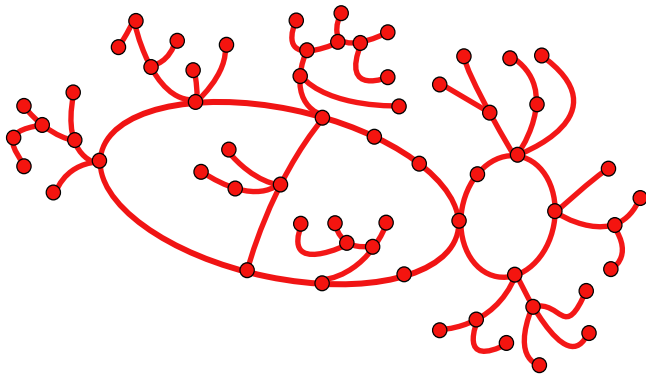


## A plot

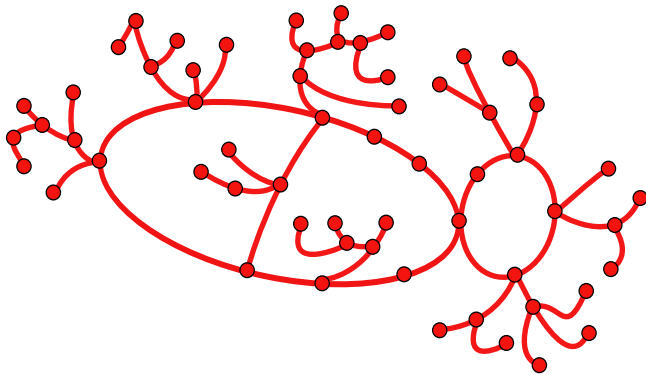


Probability curve for planar graphs and SP-graphs  
(top and bottom, respectively)

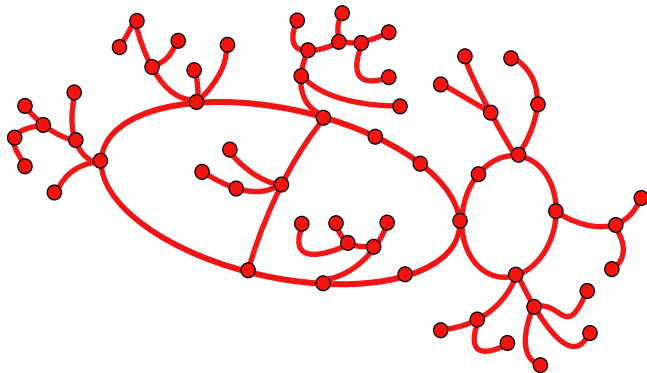
## The strategy (I): pruning a graph



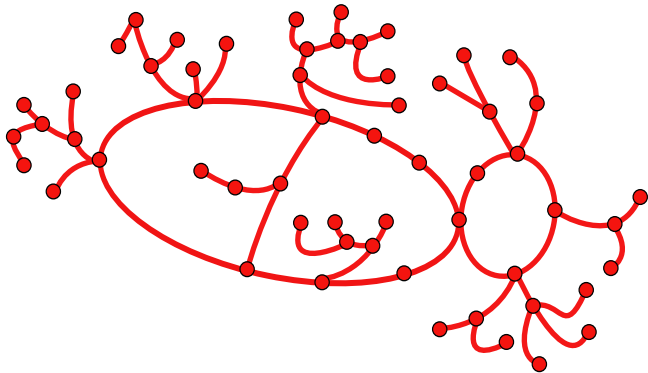
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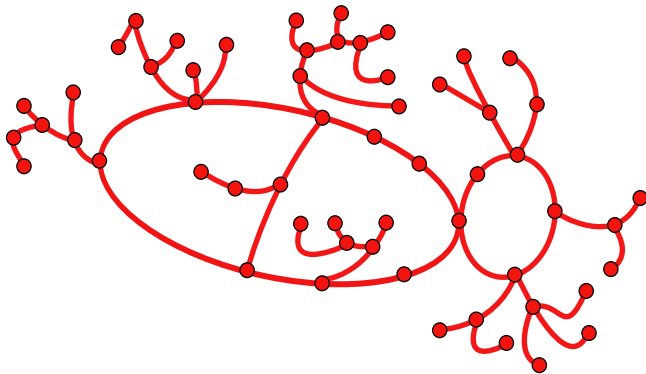
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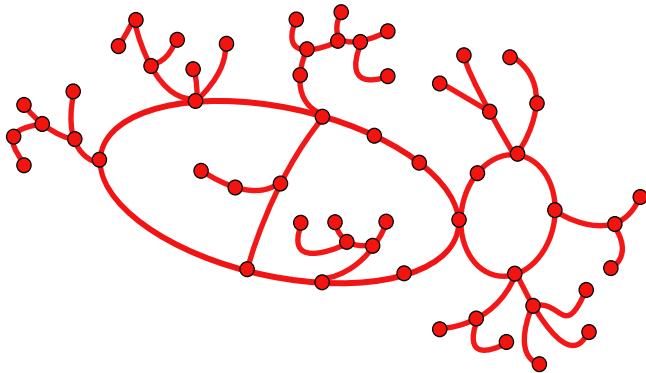
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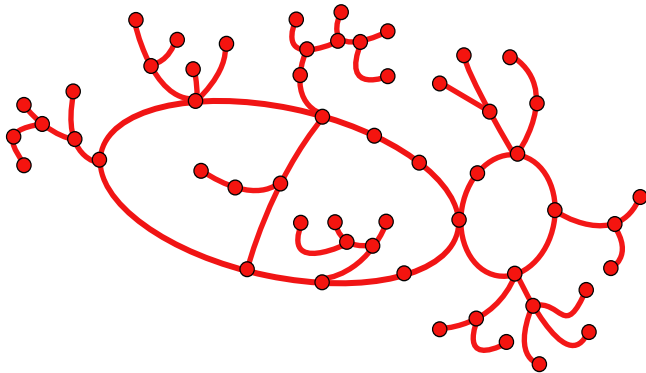
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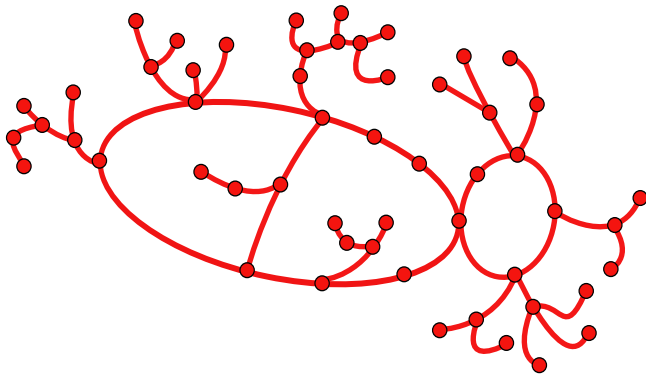


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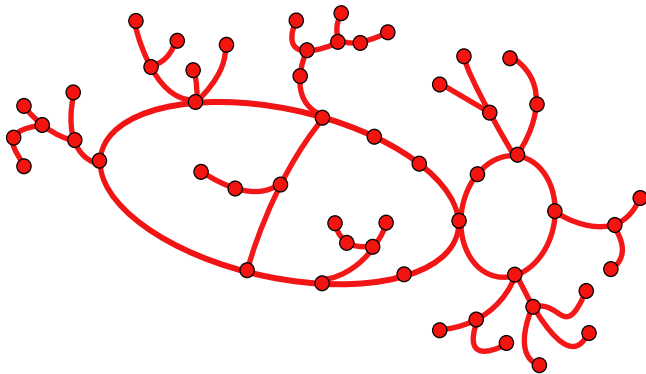




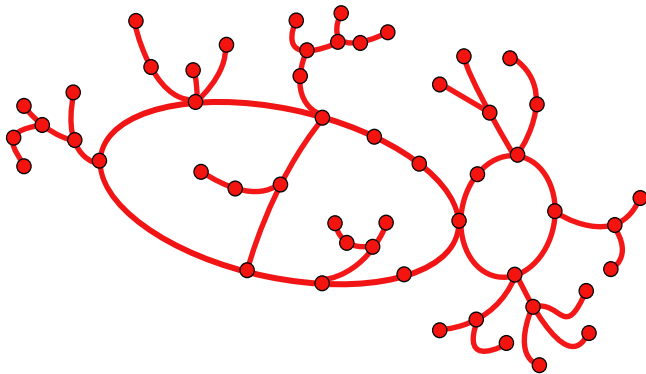
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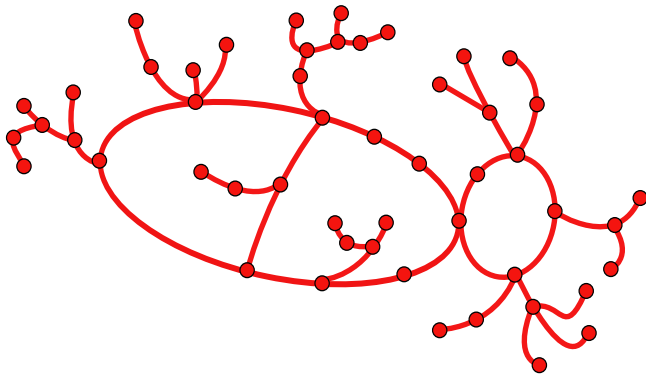
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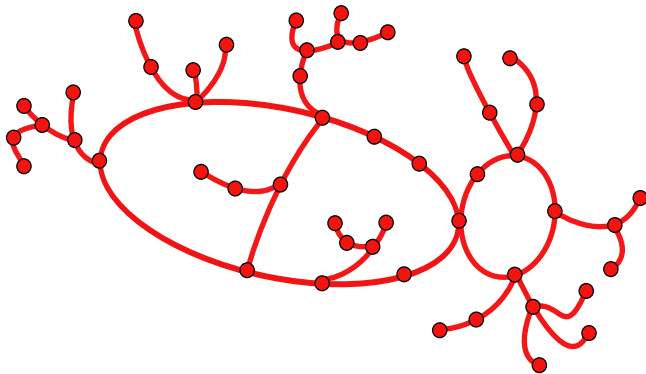
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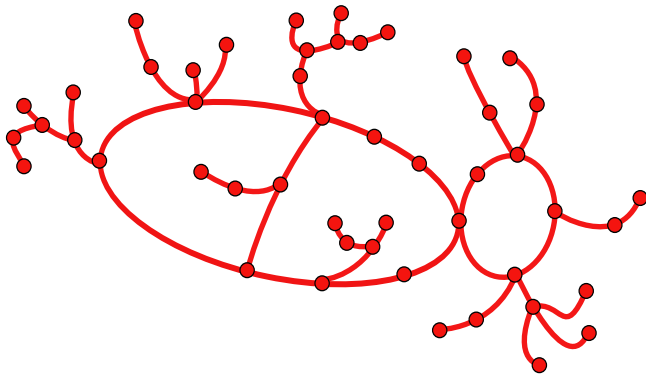
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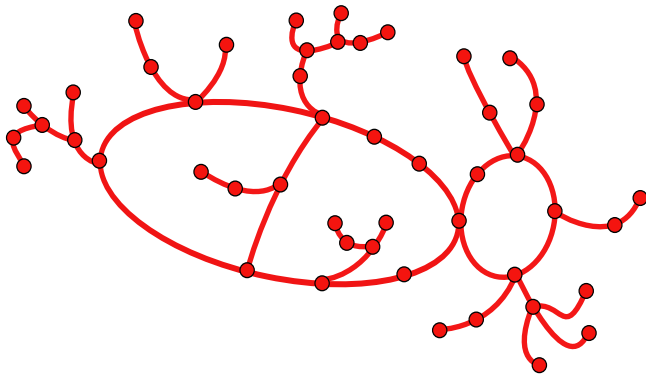
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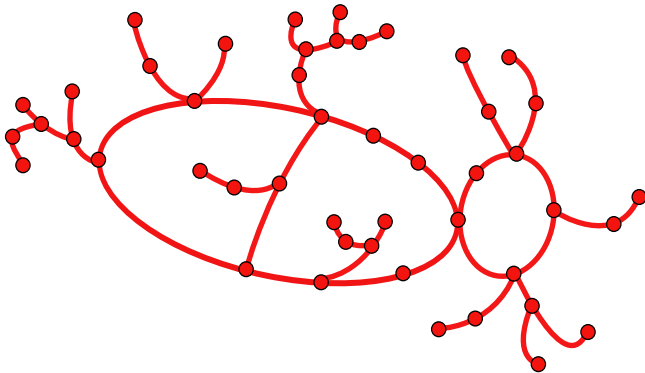
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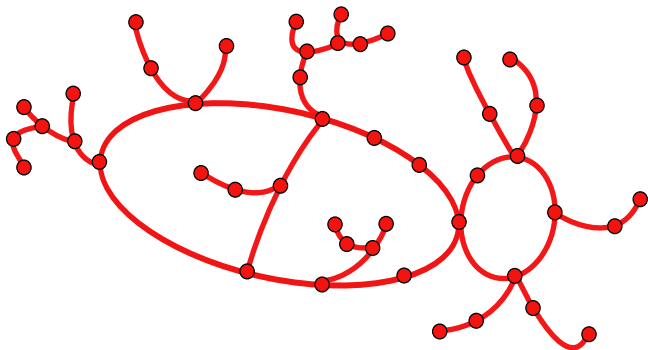


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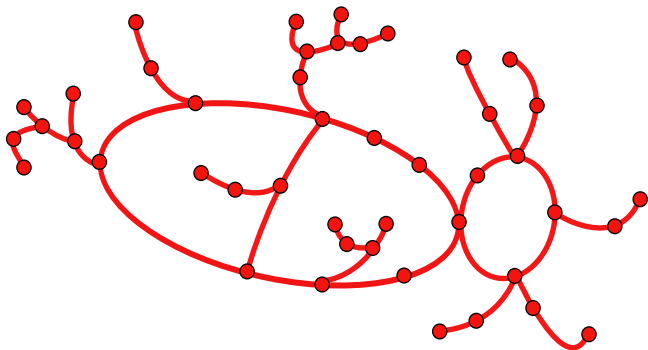




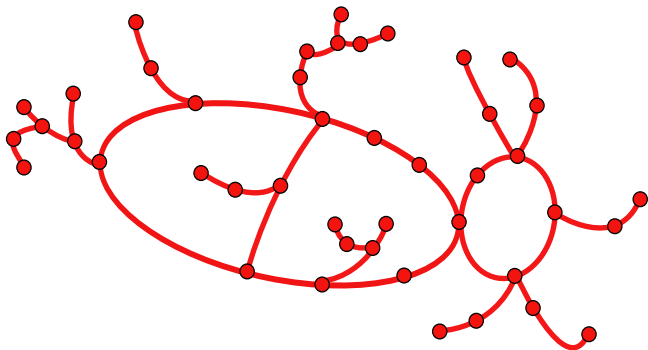
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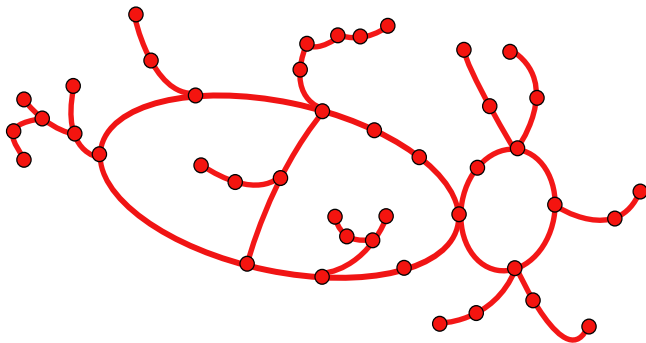
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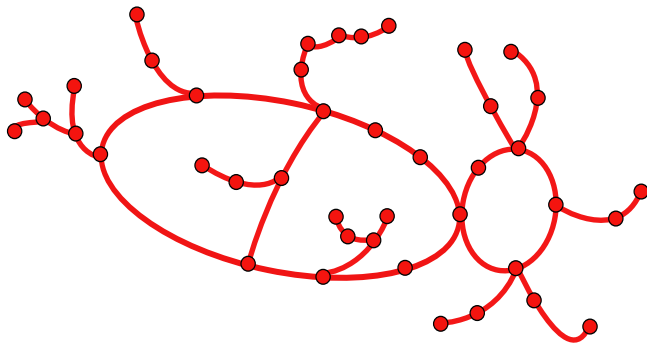
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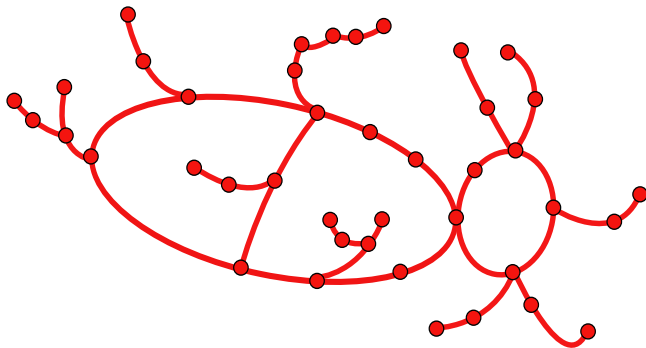
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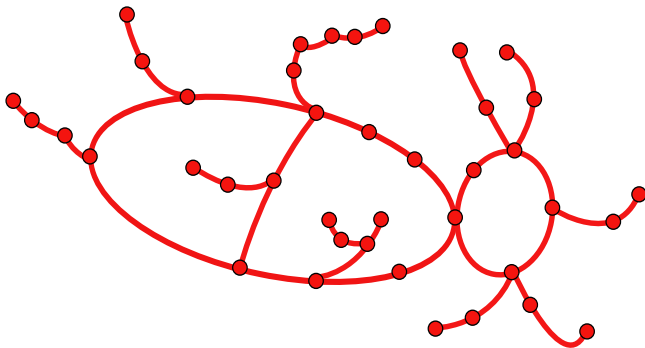
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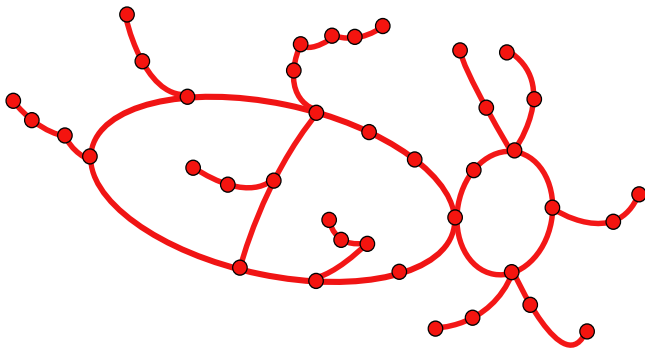
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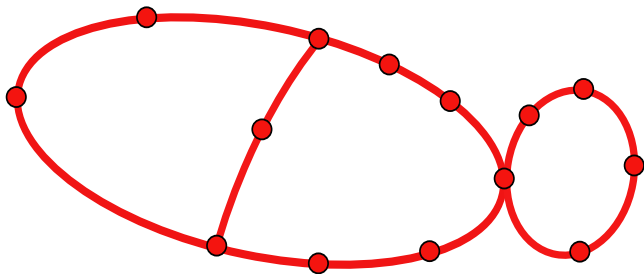


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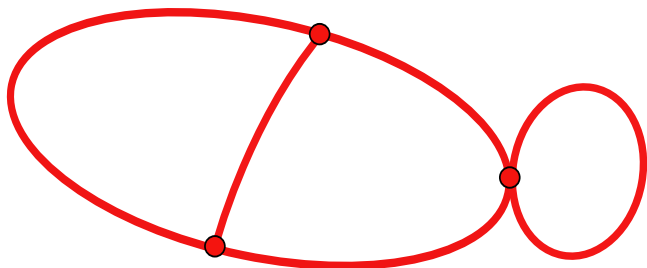




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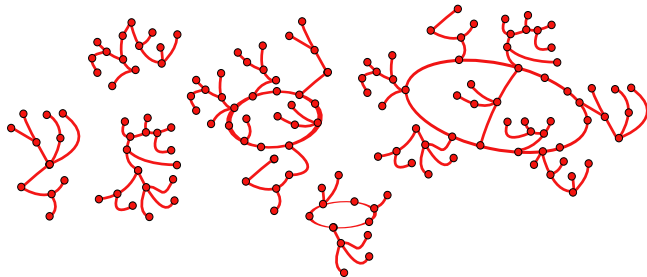


The resulting *multigraph* is the **core** of the initial graph

# The strategy (and II): shape in the critical window

Łuczak, Pittel, Wierman (1994):

the structure of a random graph in the critical window

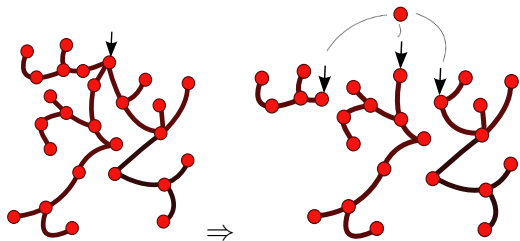


$$p(\lambda) = \frac{\text{number of planar graphs with } \frac{n}{2}(1 + \lambda n^{-1/3}) \text{ edges}}{\binom{\frac{n}{2}(1 + \lambda n^{-1/3})}{\binom{n}{2}}}$$

Hence... **We need to count!**

# Trees

We apply the previous grammar to count *rooted* trees



$$\mathcal{T} = \bullet \times \text{Set}(\mathcal{T}) \rightarrow T(x) = xe^{T(x)}$$

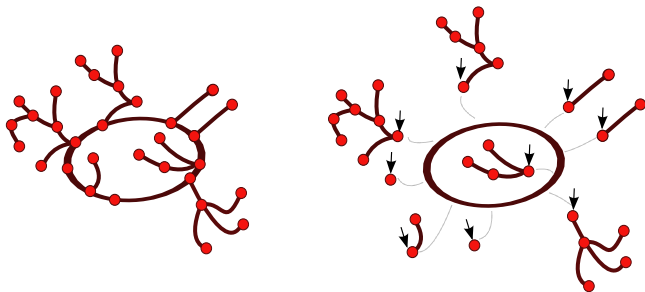
To forget the root, we just integrate:  $(xU'(x) = T(x))$

$$\int_0^x \frac{T(s)}{s} ds = \left\{ \begin{array}{l} T(s) = u \\ T'(s) ds = du \end{array} \right\} = \int_{T(0)}^{T(x)} 1-u du = T(x) - \frac{1}{2}T(x)^2$$

and the general version

$$e^{U(x)} = e^{T(x)} e^{-\frac{1}{2}T(x)^2}$$

## Unicyclic graphs

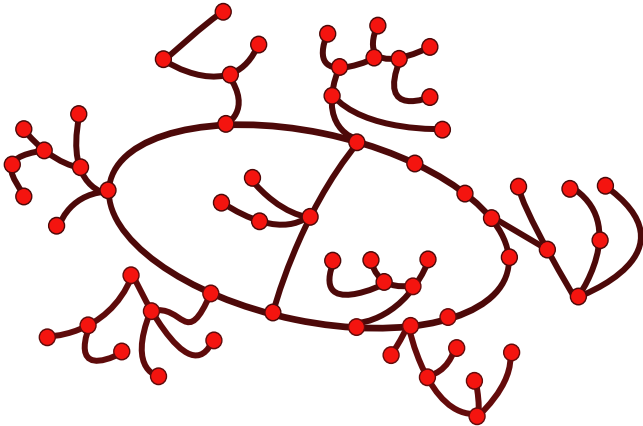


$$\mathcal{V} = \bigcirc_{\geq 3}(\mathcal{T}) \rightarrow V(x) = \sum_{n=3}^{\infty} \frac{1}{2} \frac{(n-1)!}{n!} (T(x))^n$$

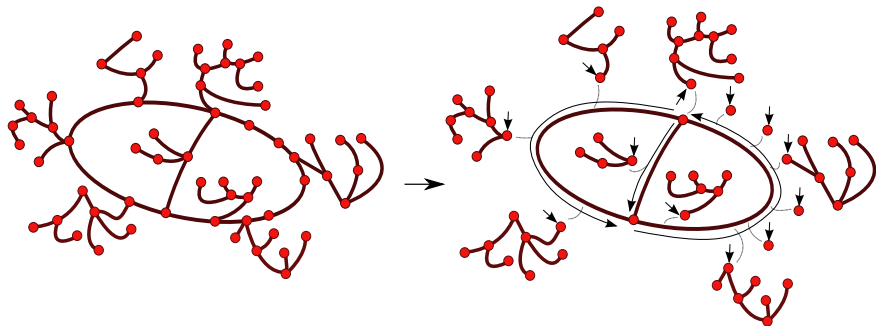
We can write  $V(x)$  in a compact way:

$$\frac{1}{2} \left( -\log(1 - T(x)) - T(x) - \frac{T(x)^2}{2} \right) \rightarrow e^{V(x)} = \frac{e^{-T(x)/2 - T(x)^2/4}}{\sqrt{1 - T(x)}}.$$

# Cubic planar multigraphs



# Planar graphs arising from cubic multigraphs



In an informal way:

$$\mathcal{G}(\bullet \leftarrow \mathcal{T}, \bullet - \bullet \leftarrow \text{Seq}(\mathcal{T}))$$

## Weighted planar cubic multigraphs

Cubic multigraphs have  $2r$  vertices and  $3r$  edges (Euler's Relation)

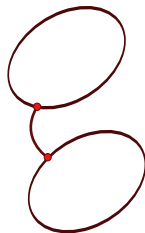
$$G(x, y) = \sum_{r \geq 1} \frac{g_r(2r)!}{(2r)!} x^{2r} y^{3r} = g(x^2 y^3)$$

We need to remember the number of loops and the number of multiple edges to avoid symmetries:

**weights**  $2^{-f_1 - f_2} (3!)^{-f_3}$



$$\frac{1}{2!} \frac{1}{6} x^2 y^3$$

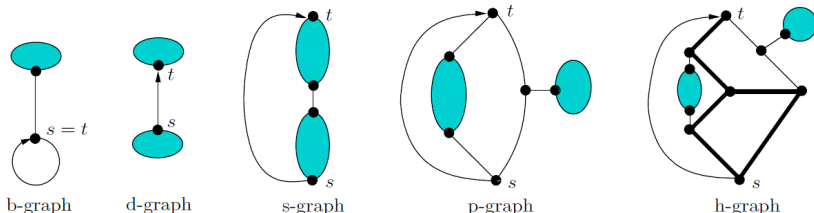


$$\frac{1}{2!} \frac{1}{2^2} x^2 y^3$$



# The decomposition

- ▶ We consider *rooted* multigraphs (namely, an edge is oriented).
- ▶ *Rooted* cubic planar multigraphs have the following form:



(From Bodirsky, Kang, Löffler, McDiarmid *Random Cubic Planar Graphs*)

## The equations

We can relate different families of rooted cubic planar graphs between them:

$$G(z) = \exp G_1(z)$$

$$3z \frac{dG_1(z)}{dz} = D(z) + C(z)$$

$$B(z) = \frac{z^2}{2}(D(z) + C(z)) + \frac{z^2}{2}$$

$$C(z) = S(z) + P(z) + H(z) + B(z)$$

$$D(z) = \frac{B(z)^2}{z^2}$$

$$S(z) = C(z)^2 - C(z)S(z)$$

$$P(z) = z^2 C(z) + \frac{1}{2} z^2 C(z)^2 + \frac{z^2}{2}$$

$$2(1 + C(z))H(z) = u(z)(1 - 2u(z)) - u(z)(1 - u(z))^3$$

$$z^2(C(z) + 1)^3 = u(z)(1 - u(z))^3.$$

## The equations: an appetizer

All GF obtained (except  $G(z)$ ) are *algebraic* GF; for instance:

$$\begin{aligned} &1048576 z^6 + 1034496 z^4 - 55296 z^2 + \\ &(9437184 z^6 + 6731264 z^4 - 1677312 z^2 + 55296) C + \\ &(37748736 z^6 + 18925312 z^4 - 7913472 z^2 + 470016) C^2 + \\ &(88080384 z^6 + 30127104 z^4 - 16687104 z^2 + 1622016) C^3 + \\ &(132120576 z^6 + 29935360 z^4 - 19138560 z^2 + 2928640) C^4 + \\ &(132120576 z^6 + 19314176 z^4 - 12429312 z^2 + 2981888) C^5 + \\ &(88080384 z^6 + 8112384 z^4 - 4300800 z^2 + 1720320) C^6 + \\ &(37748736 z^6 + 2097152 z^4 - 614400 z^2 + 524288) C^7 + \\ &(9437184 z^6 + 262144 z^4 + 65536) C^8 + 1048576 C^9 z^6 = 0. \end{aligned}$$

# The estimates

- ▶ The **excess** of a graph ( $ex(G)$ ) is the number of edges minus the number of vertices

$$n![z^n] \frac{\overbrace{U(z)^{n-M+r}}^{\text{Trees, } ex=-1}}{(n-M+r)!} \overbrace{\frac{e^{-T(z)/2-T(z)^2/4}}{\sqrt{1-T(z)}}}^{\text{Unicyclic, } ex=0} \overbrace{\frac{P(T(z))}{(1-T(z))^{3r}}}^{\text{Cubic, } ex=3r-2r=r}$$

where  $P(x)$  is a polynomial.

- ▶ We then apply a *sandwich* argument to get the estimates (where the  $g_r$  appear!)
- ▶ We finally use saddle point estimates.

**Other applications**

## General families of graphs

Many families of graphs admit an straightforward analysis:

**(Noy, Ravelomanana, R.)**

Let  $\mathcal{G} = \text{Ex}(H_1, \dots, H_k)$  and assume all the  $H_i$  are 3-connected. Let  $h_r(2r)!$  be the number of cubic multigraphs in  $\mathcal{G}$  with  $2r$  vertices. Then the limiting probability that the random graph  $G(n, \frac{n}{2}(1 + \lambda n^{-1/3}))$  is in  $\mathcal{G}$  is

$$p_{\mathcal{G}}(\lambda) = \sum_{r \geq 0} \sqrt{2\pi} h_r A \left( 3r + \frac{1}{2}, \lambda \right).$$

In particular, the limiting probability that  $G(n, \frac{n}{2})$  is in  $\mathcal{G}$  is

$$p_{\mathcal{G}}(0) = \sum_{r \geq 0} \sqrt{\frac{2}{3}} \left( \frac{4}{3} \right)^r h_r \frac{r!}{(2r)!}.$$

Moreover, for each  $\lambda$  we have

$$0 < p_{\mathcal{G}}(\lambda) < 1.$$

## Examples...please

Some interesting families fit in the previous scheme:

- ▶  $\text{Ex}(K_4)$ : series-parallel graphs: there are not 3-connected elements in the family!
- ▶  $\text{Ex}(K_{2,3}, K_4)$ : outerplanar graphs: need to adapt the equations for cubics.
- ▶  $\text{Ex}(K_{3,3})$ : The same limiting probability as planar...  
 $K_5$  does not appear as a core!
- ▶ Many others:  $\text{Ex}(K_{3,3}^+)$ ,  $\text{Ex}(K_5^-)$ ,  $\text{Ex}(K_2 \times K_3) \dots$

**Gràcies!**



# The probability of planarity of a random graph near the critical point

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