Non-D-finite excursions in the quarter plane

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joint work with
Kilian Raschel and Bruno Salvy

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Context: enumeration of lattice walks

- Nearest-neighbor walks in the quarter plane $\mathbb{N}^2$; admissible steps
  \[ \mathcal{S} \subseteq \{ \swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow \} \]

- $\mathcal{S}$-walks $= \text{walks in } \mathbb{N}^2 \text{ starting at } (0, 0) \text{ and using steps in } \mathcal{S}$. 

Questions:
- Given $\mathcal{S}$, what can be said about $F_{\mathcal{S}}(t; x, y)$?
  - Structure? (algebraic / D-finite)
  - Explicit form?
  - Asymptotics?

- $F_{\mathcal{S}}(t; 0, 0)$ counts $\mathcal{S}$-walks returning to the origin (excursions);
- $F_{\mathcal{S}}(t; 1, 1)$ counts $\mathcal{S}$-walks with prescribed length;
- $F_{\mathcal{S}}(t; 1, 0)$ counts $\mathcal{S}$-walks ending on the horizontal axis.
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- \( \mathcal{S}\)-walks = walks in \( \mathbb{N}^2 \) starting at \((0, 0)\) and using steps in \( \mathcal{S} \).

- \( f_\mathcal{S}(n; i, j) = \) number of \( \mathcal{S}\)-walks ending at \((i, j)\) and consisting of exactly \( n \) steps.
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- $f_{\mathcal{S}}(n; i, j) =$ number of $\mathcal{S}$-walks ending at $(i, j)$ and consisting of exactly $n$ steps. Complete generating function
  \[ F_{\mathcal{S}}(t; x, y) = \sum_{n=0}^{\infty} \left( \sum_{i,j=0}^{\infty} f_{\mathcal{S}}(n; i, j)x^i y^j \right) t^n \in \mathbb{Q}[x, y][[t]]. \]
Context: enumeration of lattice walks

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$$F_{\mathcal{S}}(t; x, y) = \sum_{n=0}^{\infty} \left( \sum_{i,j=0}^{\infty} f_{\mathcal{S}}(n; i, j)x^iy^j \right)t^n \in \mathbb{Q}[x, y][[t]].$$

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Questions: Given \( \mathcal{S} \), what can be said about \( F_{\mathcal{S}}(t; x, y) \)?
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\( F_{\mathcal{S}}(t; 0, 0) \sim \) counts \( \mathcal{S} \)-walks returning to the origin (excursions);
\( F_{\mathcal{S}}(t; 1, 1) \sim \) counts \( \mathcal{S} \)-walks with prescribed length;
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Small step sets in the quarter plane

\[ \mathcal{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\} \]

There are \(2^8\) such sets.
Small step sets in the quarter plane

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Some of these $$2^8 = 256$$ step sets are:

trivial,
Small step sets in the quarter plane

\[ S \subseteq \{ -1, 0, 1 \}^2 \setminus \{(0, 0)\} \]

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Some of these \(2^8 = 256\) step sets are:

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There are \(2^8\) such sets.

Some of these \(2^8 = 256\) step sets are:

- trivial,
- simple,
- intrinsic to the half plane,
Small step sets in the quarter plane

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Some of these \(2^8 = 256\) step sets are:

- trivial,
- simple,
- intrinsic to the half plane,
- symmetrical.

Finally, there remain 79 inherently different cases!
An example of walk with small steps

1, 0, 2, 1, 10, 14, 75, 178, 738, 2304, 8753, 31186, 117244, 444107, 1698959, 6637212, 25978768, 103470800, 413378548, ...
Two important classes of univariate power series

D-finite series

algebraic series
Two important classes of univariate power series

**D-finite**: $S(t) \in \mathbb{Q}[[t]]$ satisfying a linear differential equation with polynomial coefficients $c_r(t)S^{(r)}(t) + \cdots + c_0(t)S(t) = 0$. 

**Algebraic**: $S(t) \in \mathbb{Q}[[t]]$ root of a polynomial $P \in \mathbb{Q}[t, T]$. 

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Two important classes of univariate power series

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Two important classes of multivariate power series

$S \in \mathbb{Q}[[x, y, t]]$ is **D-finite** if the set of all partial derivatives of $S$ spans a finite-dimensional vector space over $\mathbb{Q}(x, y, t)$. 

**D-finite series**

algebraic series

$S \in \mathbb{Q}[[x, y, t]]$ is **algebraic** if it is the root of a $P \in \mathbb{Q}[x, y, t, t]$. 
Two important classes of multivariate power series

\[ S \in \mathbb{Q}[[x, y, t]] \] is \textit{D-finite} if the set of all partial derivatives of \( S \) spans a finite-dimensional vector space over \( \mathbb{Q}(x, y, t) \).

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The group of a walk: an example

The jump characteristic polynomial $x + \frac{1}{x} + y + \frac{1}{y}$
The group of a walk: an example

The jump characteristic polynomial \( x + \frac{1}{x} + y + \frac{1}{y} \) is left invariant under

\[
\psi(x, y) = \left( x, \frac{1}{y} \right), \quad \phi(x, y) = \left( \frac{1}{x}, y \right),
\]
The group of a walk: an example

The jump characteristic polynomial $x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$\psi(x, y) = \left(x, \frac{1}{y}\right)$, \hspace{1cm} $\phi(x, y) = \left(\frac{1}{x}, y\right)$,

and thus under any element of the group

$\langle \psi, \phi \rangle = \left\{ (x, y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}$. 
The group of a walk: the general case

The jump polynomial

\[
\sum_{(i,j) \in \mathcal{G}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j
\]
The group of a walk: the general case

The jump polynomial

\[ \sum_{(i,j) \in \mathcal{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j \]

is left invariant under

\[ \psi(x, y) = \left( x, \frac{A_{-1}(x) 1}{A_{+1}(x) y} \right), \quad \phi(x, y) = \left( \frac{B_{-1}(y) 1}{B_{+1}(y) x}, y \right), \]
The group of a walk: the general case

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and thus under any element of the group

\[ \mathcal{G}_\mathcal{S} := \langle \psi, \phi \rangle. \]
Some examples

Order 4,
Some examples

Order 4,

order 6,
Some examples

Order 4,

order 6,

order 8,
Some examples

Order 4, order 6, order 8, order $\infty$.
The 79 cases: finite and infinite groups

79 step sets
The 79 cases: finite and infinite groups

79 step sets

23 admit a finite group
[Mishna’07]

56 have an infinite group
[Bousquet-Mélou-Mishna’10]

• [Mishna-Rechnitzer’07] and [Melczer-Mishna’12] for 5 singular cases
• [Kurkova-Raschel’12] and [B.-Raschel-Salvy’12] for all others
The 79 cases: finite and infinite groups

- 79 step sets
- 23 admit a finite group
  - [Mishna’07]
- 56 have an infinite group
  - [Bousquet-Mélou-Mishna’10]
- All are D-finite
- 19 transcendental
  - [Gessel-Zeilberger’92]
  - [Bousquet-Mélou’02]
- 4 are algebraic
  - (3 Kreweras-type + Gessel)
  - [B.-Kauers’10]

- All non-D-finite
- • [Mishna-Rechnitzer’07] and [Melczer-Mishna’12] for 5 singular cases
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This talk 10/26
The 79 cases: finite and infinite groups

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The 23 cases with a finite group

(i) 16 with a vertical symmetry, and group isomorphic to $D_2$

(ii) 5 with a diagonal or an anti-diagonal symmetry, and group isomorphic to $D_3$

(iii) 2 with group isomorphic to $D_4$

In red, cases with algebraic generating functions $F_G(t; x, y)$
The 56 cases with infinite group

In red, singular cases, solved by [Melczer & Mishna 2012]

In blue, non-singular cases – this talk
Main techniques for proving non D-finiteness

(1) Infinite number of singularities
   [Bousquet-Mélou & Petkovsek’03 Mishna & Rechnitzer’07, Melczer & Mishna’12, Kurkova & Raschel’12]

(2) Lacunarity
   [Bousquet-Mélou & Petkovsek’03]

(3) Asymptotics + Number theory: This talk
Main techniques for proving non D-finiteness

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(3) Asymptotics + Number theory: This talk

▷ (3) is fully algorithmic.
A historical example

Knight walks: $\mathcal{G} = \{(-1, 2), (2, -1)\}$

$$a_{m,n} = \begin{cases} 
0 & \text{if } m < 0 \text{ or } n < 0 \\
1 & \text{if } m = n = 1 \\
a_{m+1,n-2} + a_{m-2,n+1} & \text{otherwise}
\end{cases}$$

*Theorem* [Bousquet-Mélou & Petkovsek’03] The generating series $F(x, y) = \sum_{m,n\geq0} a_{m,n}x^my^n$ is not D-finite.

▷ Key argument: $F(x, 0)$ has infinitely many singularities.
Main result

Theorem [B., Raschel & Salvy 2012] Let $\mathcal{S}$ be one of the 74 non-singular step sets. The following assertions are equivalent:

1. The full generating series $F_{\mathcal{S}}(t; x, y)$ is D-finite
2. the excursions generating series $F_{\mathcal{S}}(t; 0, 0)$ is D-finite
3. the excursions sequence $[t^n] F_{\mathcal{S}}(t; 0, 0)$ is $\sim K \cdot \rho^n \cdot n^\alpha$, with $\alpha \in \mathbb{Q}$
4. the group $\mathcal{G}_\mathcal{S}$ is finite.
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(4) $\Rightarrow$ (1) [Bousquet-Mélou & Mishna 2010] $+$ [B. & Kauers 2010]
Main result

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4. the group $\mathcal{G}$ is finite.

$(4) \Rightarrow (1)$ [Bousquet-Mélou & Mishna 2010] + [B. & Kauers 2010]

$(1) \Rightarrow (2)$ $F_{\mathcal{G}}(t; 0, 0)$ is the Hadamard product of $F_{\mathcal{G}}(t; x, y)$ and $\frac{1}{1-t}$, and Hadamard product preserves D-finiteness [Lipshitz 1988]
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$(2) \Rightarrow (3)$ and $(3) \Rightarrow (4)$ This talk
Main result: algorithmic proof of non-D-finiteness

Theorem [B.-Rachel-Salvy, 2012]

Let $\mathcal{S}$ be one of the 51 step sets with infinite group $G_{\mathcal{S}}$, and such that the excursions series $F_{\mathcal{S}}(t; 0, 0)$ is not equal to 1. Then $F_{\mathcal{S}}(t; 0, 0)$, and in particular $F_{\mathcal{S}}(t; x, y)$, are non-D-finite.
Main result: algorithmic proof of non-D-finiteness

*Theorem [B.-Rachel-Salvy, 2012]*

Let $\mathcal{S}$ be one of the 51 step sets with infinite group $\mathcal{G_0}$, and such that the excursions series $F_\mathcal{G}(t;0,0)$ is not equal to 1. Then $F_\mathcal{G}(t;0,0)$, and in particular $F_\mathcal{G}(t;x,y)$, are non-D-finite.

▶ Algorithmic, computer-driven, proof. Uses Gröbner basis computations, polynomial factorization, cyclotomy testing.
▶ Based on two ingredients: asymptotics + irrationality.

▶ [Kurkova & Raschel 2012] Alternative proof of $F_\mathcal{G}(t;x,y)$ is non-D-finite. No human proof yet for $F_\mathcal{G}(t;0,0)$ non-D-finite.
First ingredient: probability meets combinatorics

[Denisov & Wachtel 2011] For very general walks in cones in $\mathbb{N}^d$:

$$[t^n] F_{\mathcal{G}}(t; 0, 0) \sim K \cdot \rho^n \cdot n^{-p-d/2}, \quad \text{where}$$
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$\rho$ is an algebraic number depending on $\mathcal{G}$.
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where

- $\rho$ is an algebraic number depending on $\mathcal{G}$
- $p = \sqrt{\lambda + (d/2 - 1)^2} - (d/2 - 1)$, where $\lambda > 0$ is the least Dirichlet eigenvalue of the boundary problem

$$\begin{cases}
\nabla u(x) = -\lambda u(x) \text{ for } x \in \Omega, \\
\text{and } u|_{\partial\Omega} = 0,
\end{cases}$$

where $\nabla$ is the (spherical) Laplacian, and $\Omega \subset S^{d-1}$ is a spherical polygon depending on $\mathcal{G}$.
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where $\nabla$ is the (spherical) Laplacian, and $\Omega \subset S^{d-1}$ is a spherical polygon depending on $\mathbb{G}$

Let $\mathcal{G}$ be one of the 51 step sets in $\mathbb{N}^2$ with infinite group $\mathcal{G}_\mathcal{G}$, and such that the excursions series $F_{\mathcal{G}}(t; 0, 0)$ is not equal to 1. Let $\chi$ be the characteristic polynomial of $\mathcal{G}$. Then the system

$$\frac{\partial \chi}{\partial x} = \frac{\partial \chi}{\partial y} = 0$$

has a unique solution $(x_0, y_0) \in \mathbb{R}^2_{>0}$, and

$$[t^n] F_{\mathcal{G}}(t; 0, 0) \sim K \cdot \rho^n \cdot n^\alpha,$$

where

- $\rho$ is the algebraic number $\rho = \chi(x_0, y_0)$,
- $\alpha = \pi / \arccos(c) - 1$, where $c < 0$ is the algebraic number

$$c = \frac{\frac{\partial^2 \chi}{\partial x \partial y}}{\sqrt{\frac{\partial^2 \chi}{\partial x^2} \cdot \frac{\partial^2 \chi}{\partial y^2}}}(x_0, y_0).$$

Proof uses explicit resolution of the 2D boundary problem.
Corollary B

Let \((a_n)_{n \geq 0}\) be an integer-valued sequence whose \(n\)-th term \(a_n\) behaves asymptotically like \(K \cdot \rho^n \cdot n^\alpha\), for some real \(K > 0\). If the growth constant \(\rho\) is transcendental, or if the singular exponent \(\alpha\) is irrational, then the generating series \(\sum_{n \geq 0} a_n t^n\) is not D-finite.
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▷ The weaker conclusion that \(\sum_{n \geq 0} a_n t^n\) is not algebraic is a consequence of the Newton-Puiseux theorem and of transfer theorems based on Cauchy’s integral formula [Flajolet’80]
Second ingredient: number theory meets combinatorics

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▷ If the assumption \(\alpha\) irrational were replaced by \(\alpha\) transcendental, the result would follow by [Birkhoff-Trjitzinsky’32, Turrittin’55]
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- Corollary B is a direct consequence of deep arithmetic results about G-functions [Chudnovsky’85, André’89, Katz’70]
To conclude the non-D-finiteness of \( F_{\mathcal{G}}(t; 0, 0) \), it suffices to prove that \( \arccos(c)/\pi \) is irrational, where

\[
c = \frac{\frac{\partial^2 \chi}{\partial x \partial y}(x_0, y_0)}{\sqrt{\frac{\partial^2 \chi}{\partial x^2} \cdot \frac{\partial^2 \chi}{\partial y^2}}}(x_0, y_0), \quad \text{with} \quad \frac{\partial \chi}{\partial x}(x_0, y_0) = \frac{\partial \chi}{\partial y}(x_0, y_0) = 0,
\]

and

\[
\chi(x, y) = \sum_{(i,j) \in \mathcal{G}} x^i y^j \in \mathbb{Q}[x, x^{-1}, y, y^{-1}].
\]
Algorithmic blending of probability and arithmetics

To conclude the non-D-finiteness of $F_{\mathcal{G}}(t; 0, 0)$, it suffices to prove that $\arccos(c)/\pi$ is irrational, where

$$c = \frac{\frac{\partial^2 \chi}{\partial x \partial y}(x_0, y_0)}{\sqrt{\frac{\partial^2 \chi}{\partial x^2} \cdot \frac{\partial^2 \chi}{\partial y^2}}}$$

and

$$\chi(x, y) = \sum_{(i,j) \in \mathcal{G}} x^i y^j \in \mathbb{Q}[x, x^{-1}, y, y^{-1}].$$

This can be done \textit{algorithmically} in two main steps:

(S1) determine the minimal polynomial, $\mu_c$, of $c$

(S2) prove that the numerator of $\mu_c \left( \frac{x^2 + 1}{2x} \right)$ contains no cyclotomic polynomial factor.
Step (S1)

▷ An algebraic equation for the exponential growth $\rho$ is obtained by eliminating $x$ and $y$ from

$$\frac{\partial \chi}{\partial x} = 0, \quad \frac{\partial \chi}{\partial y} = 0, \quad \rho - \chi = 0.$$ 

▷ This elimination is a routine task in effective algebraic geometry.

▷ It is performed using Gröbner bases and polynomial factorization.
An algebraic equation for the exponential growth $\rho$ is obtained by eliminating $x$ and $y$ from

$$\frac{\partial \chi}{\partial x} = 0, \quad \frac{\partial \chi}{\partial y} = 0, \quad \rho - \chi = 0.$$ 

This elimination is a routine task in effective algebraic geometry. It is performed using Gröbner bases and polynomial factorization. $\rho = \chi(x_0, y_0) = \frac{P}{Q}(x_0, y_0)$ is a root of a generator of $I \cap \mathbb{Q}[t]$, for

$$I = \left\langle \text{num} \left( \frac{\partial \chi}{\partial x} \right), \ \text{num} \left( \frac{\partial \chi}{\partial y} \right), \ P(x, y) - t \cdot Q(x, y) \right\rangle.$$ 

Generator obtained by a Gröbner basis computation for $I$, using an elimination order with $(x, y) > t.$
Similarly, an algebraic equation for the correlation coefficient $c$ is obtained by eliminating $x$ and $y$ from

$$\frac{\partial \chi}{\partial x} = 0, \quad \frac{\partial \chi}{\partial y} = 0, \quad c^2 - \left(\frac{\partial^2 \chi}{\partial x \partial y}\right)^2 \left(\frac{\partial^2 \chi}{\partial x^2} \cdot \frac{\partial^2 \chi}{\partial y^2}\right) = 0.$$
**Step (S2): arccos(c)/π is irrational**

\[ \text{arccos}(c)/\pi \text{ is rational } \iff c = (x + 1/x)/2 \text{ with } x \text{ a root of unity} \]

\[ \iff \exists \mu_c, R(x) = x^{\deg \mu_c} \cdot \mu_c\left(\frac{x^2 + 1}{2x}\right) \text{ is divisible by a cyclotomic polynomial} \]

**Our algorithm:** In all the 51 cases, \( R(x) \) is irreducible, it has degree \( \leq 28 \), and at least one coefficient of absolute value greater than 3.

[Lehmer'36, Rosser & Schoenfeld'62]: if a cyclotomic polynomial has degree \( \leq 30 \), then its coefficients belong to \( \{-2, -1, 0, 1, 2\} \)

▷ In the 51 cases, \( \text{arccos}(c)/\pi \), and thus also \( \alpha \), is irrational, q.e.d.
The algorithm on the example

\[ S = \star \]

\[ S := \{[-1,0],[0,1],[1,0],[1,-1],[0,-1]\} :\]
\[ \chi := \frac{1}{x} + \frac{1}{y} + x + y + \frac{x}{y} \]
\[ \chi_x := x^2 + x^2 y - y, \quad \chi_y := y^2 - x - 1. \]

\[ G := \text{Groebner}[\text{Basis}](\{\chi_x, \chi_y, \text{n numer}(t^2 - \frac{\text{diff}(\chi, x)^2}{\text{diff}(\chi, x, x)/\text{diff}(\chi, y, y)})\}, \text{lexdeg}([x, y], [t])) :\]
\[ p := \text{factor}(\text{op}(\text{remove}(\text{has}, G, \{x, y\}))) ; \]
\[ p := (4 t^2 + 1)(8 t^3 + 8 t^2 + 6 t + 1)(8 t^3 - 8 t^2 + 6 t - 1). \]

This polynomial has only two real roots, ±c. The sign condition c < 0 identifies the minimal polynomial of c as \( \mu_c = 8 t^3 + 8 t^2 + 6 t + 1 \)

\[ \text{mu}_c := 8 * t^3 + 8 * t^2 + 6 * t + 1 ; \]
\[ R := \text{expand}(x^3 \ast \text{subs}(t = (x^2 + 1)/x/2, \text{mu}_c), \text{sort}) ; \]
\[ R(x) = x^6 + 2 x^5 + 6 x^4 + 5 x^3 + 6 x^2 + 2 x + 1. \]

\[ \text{irreduc}(R), \text{numtheory}[\text{iscyclotomic}](R, x) ; \]
\[ \text{true, false} \]
Summary and future work

- algorithmic proof of non-D-finiteness of $F(t; 0, 0)$ and $F(t; x, y)$ for the 51 non-singular cases with infinite group
- 2D classification of $F(t; 0, 0)$ and $F(t; x, y)$ is fully completed
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😊 lack of “purely human” proofs of non-D-finiteness of $F(t; 0, 0)$
😊 still missing an enlightening proof of: finite group $↔$ D-finite

Extensions

1. Longer 2D steps [B., Bousquet-Mélou & Melczer, in progress]
   • 680 step sets with one large step, 643 proven non D-finite
   • 5910 step sets with two large steps, 5754 proven non D-finite

2. 3D walks [B., Raschel, Salvy, in progress]
   • New phenomenon (empirically discovered, no proof yet):
     $\exists$ step sets (3D Kreweras) with finite group and non-D-finite GF!

Open questions

1. Prove directly the needed arithmetic and combinatorial results.
2. Are the singular exponents $\alpha$ even transcendental?
3. Nature of $F(t; 1, 1)$ in the 51 non-singular cases?
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Thanks for your attention!