

# Non-D-finite excursions in the quarter plane

Alin Bostan



joint work with  
Kilian Raschel and Bruno Salvy

ALEA 2013,  
March 18, 2013

## Context: enumeration of lattice walks

- ▷ *Nearest-neighbor walks in the quarter plane*  $\mathbb{N}^2$ ; admissible steps

$$\mathfrak{S} \subseteq \{ \swarrow, \leftarrow, \nearrow, \uparrow, \searrow, \rightarrow, \downarrow \}.$$

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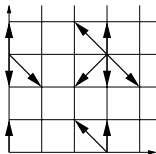
$F_{\mathfrak{S}}(t; 0, 0) \rightsquigarrow$  counts  $\mathfrak{S}$ -walks returning to the origin (**excursions**);

$F_{\mathfrak{S}}(t; 1, 1) \rightsquigarrow$  counts  $\mathfrak{S}$ -walks with prescribed length;

$F_{\mathfrak{S}}(t; 1, 0) \rightsquigarrow$  counts  $\mathfrak{S}$ -walks ending on the horizontal axis.

## Small step sets in the quarter plane

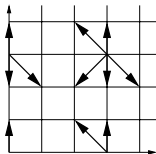
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There are  $2^8$  such sets.

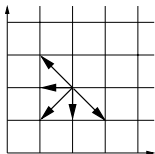
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Some of these  $2^8 = 256$  step sets are:

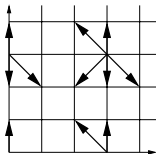


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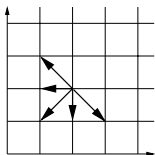
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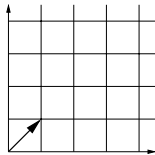


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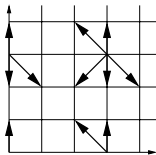
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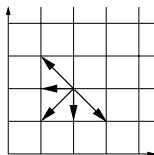
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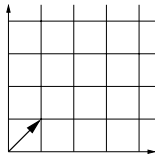


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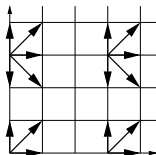
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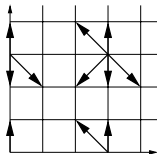
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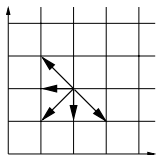
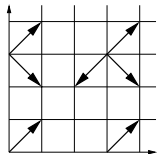
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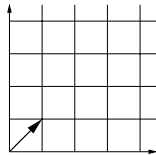


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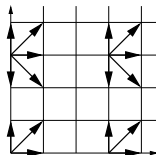
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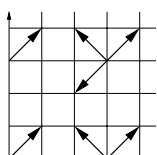
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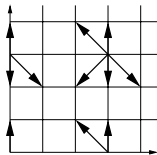
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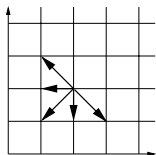
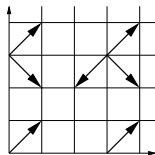
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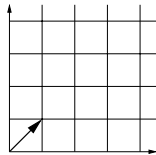


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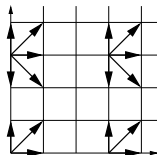
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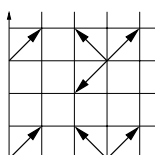
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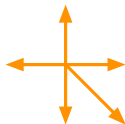
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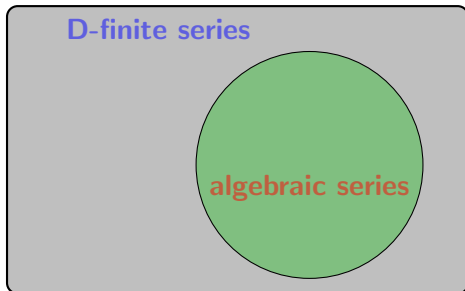
Finally, there remain 79 inherently different cases!

## An example of walk with small steps



1, 0, 2, 1, 10, 14, 75, 178, 738, 2304, 8753, 31186, 117244,  
444107, 1698959, 6637212, 25978768, 103470800, 413378548, ...

## Two important classes of univariate power series



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*D-finite*:  $S(t) \in \mathbb{Q}[[t]]$  satisfying a linear differential equation with polynomial coefficients  $c_r(t)S^{(r)}(t) + \cdots + c_0(t)S(t) = 0$ .

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*Algebraic*:  $S(t) \in \mathbb{Q}[[t]]$  root of a polynomial  $P \in \mathbb{Q}[t, T]$ .



## Two important classes of multivariate power series



$S \in \mathbb{Q}[[x, y, t]]$  is *D-finite* if the set of all partial derivatives of  $S$  spans a finite-dimensional vector space over  $\mathbb{Q}(x, y, t)$ .

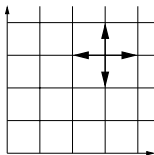
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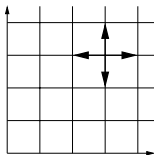
$S \in \mathbb{Q}[[x, y, t]]$  is *algebraic* if it is the root of a  $P \in \mathbb{Q}[x, y, t, T]$ .

## The group of a walk: an example



The jump characteristic polynomial  $x + \frac{1}{x} + y + \frac{1}{y}$

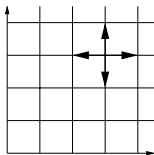
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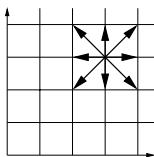
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and thus under any element of the group

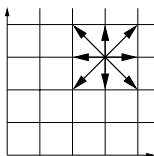
$$\langle \psi, \phi \rangle = \left\{ (x, y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}.$$

## The group of a walk: the general case



The jump polynomial  $\sum_{(i,j) \in \mathcal{G}} x^i y^j = \sum_{i=-1}^1 B_i(y) x^i = \sum_{j=-1}^1 A_j(x) y^j$

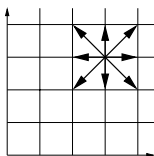
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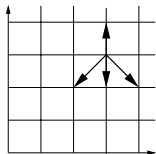
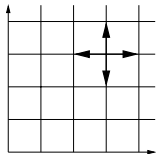
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$$\mathcal{G}_{\mathcal{G}} := \langle \psi, \phi \rangle.$$

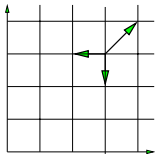
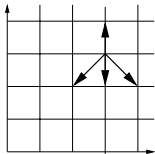
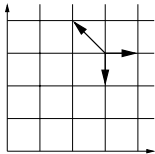
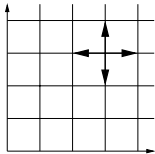


## Some examples



Order 4,

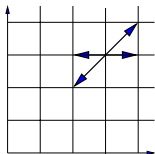
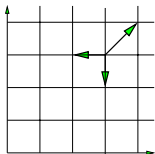
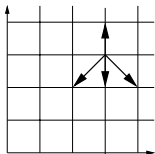
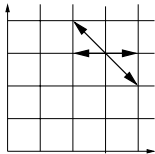
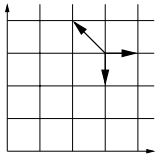
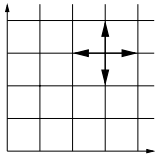
## Some examples



Order 4,

order 6,

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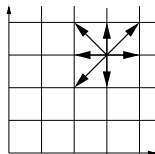
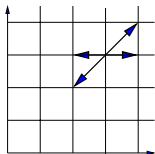
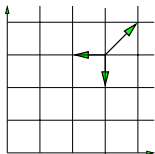
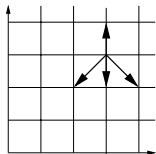
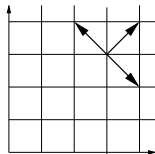
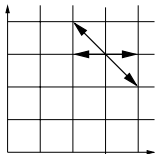
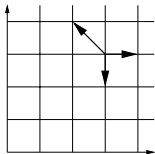
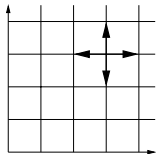


Order 4,

order 6,

order 8,

## Some examples



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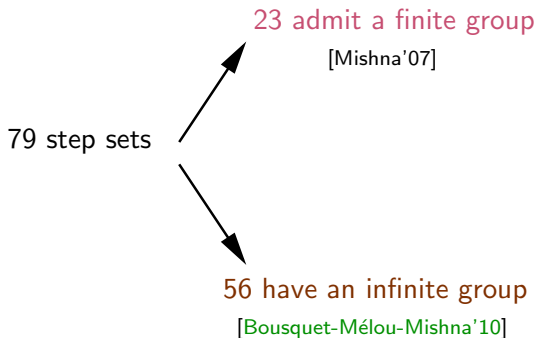
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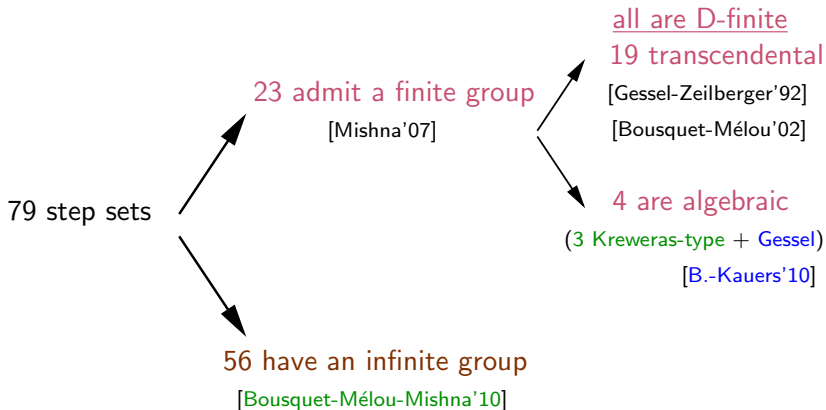
## The 79 cases: finite and infinite groups

79 step sets

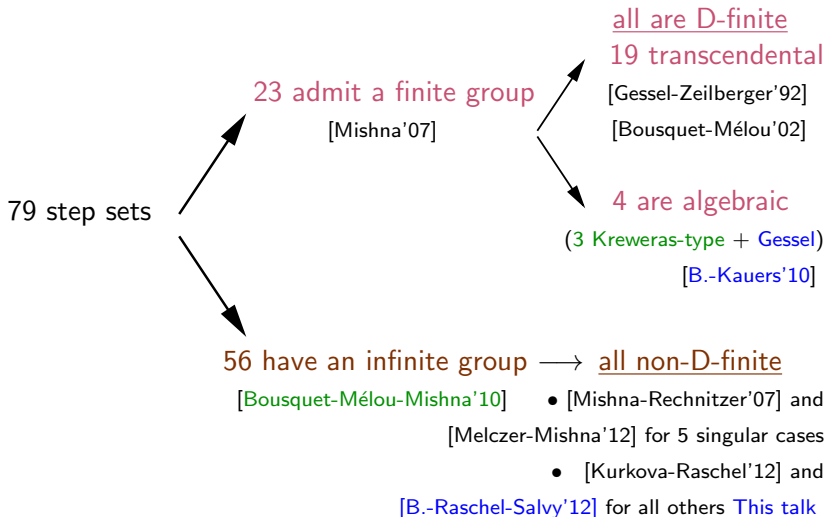
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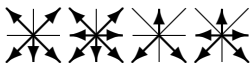
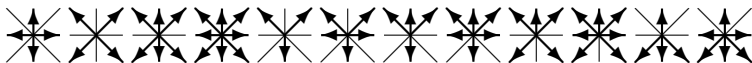
## The 79 cases: finite and infinite groups





## The 23 cases with a finite group

- (i) 16 with a vertical symmetry, and group isomorphic to  $D_2$



- (ii) 5 with a diagonal or an anti-diagonal symmetry, and group isomorphic to  $D_3$

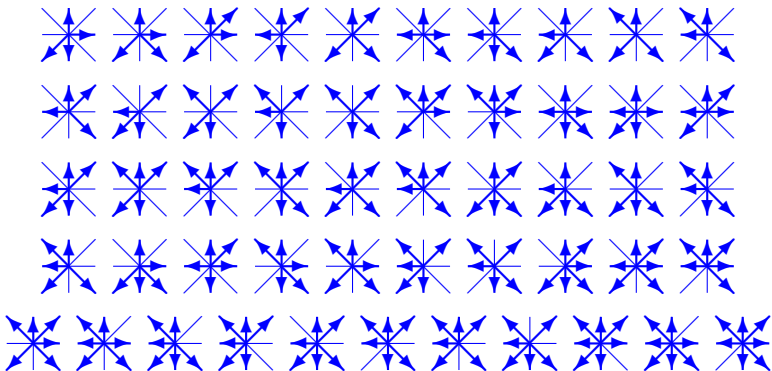


- (iii) 2 with group isomorphic to  $D_4$



In red, cases with algebraic generating functions  $F_{\mathcal{G}}(t; x, y)$

## The 56 cases with infinite group



In red, singular cases, solved by [Melczer & Mishna 2012]

In blue, non-singular cases – this talk

## Main techniques for proving non D-finiteness

- (1) Infinite number of singularities  
[Bousquet-Mélou & Petkovsek'03 Mishna & Rechnitzer'07,  
Melczer & Mishna'12, Kurkova & Raschel'12]
- (2) Lacunarity  
[Bousquet-Mélou & Petkovsek'03]
- (3) Asymptotics + Number theory: **This talk**

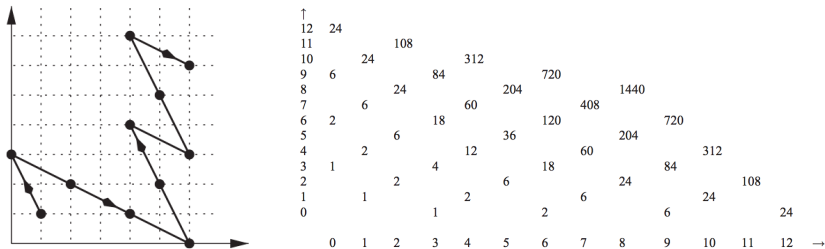
## Main techniques for proving non D-finiteness

- (1) Infinite number of singularities  
[Bousquet-Mélou & Petkovsek'03 Mishna & Rechnitzer'07,  
Melczer & Mishna'12, Kurkova & Raschel'12]
  - (2) Lacunarity  
[Bousquet-Mélou & Petkovsek'03]
  - (3) Asymptotics + Number theory: **This talk**
- ▷ (3) is fully algorithmic.

## A historical example

Knight walks:  $\mathfrak{S} = \{(-1, 2), (2, -1)\}$

$$a_{m,n} = \begin{cases} 0 & \text{if } m < 0 \text{ or } n < 0 \\ 1 & \text{if } m = n = 1 \\ a_{m+1,n-2} + a_{m-2,n+1} & \text{otherwise} \end{cases}$$



*Theorem* [Bousquet-Mélou & Petkovsek'03] The generating series

$$F(x, y) = \sum_{m, n \geq 0} a_{m,n} x^m y^n \text{ is not D-finite.}$$

▷ Key argument:  $F(x, 0)$  has **infinitely many singularities**.

## Main result

*Theorem* [B., Raschel & Salvy 2012] Let  $\mathfrak{S}$  be one of the 74 non-singular step sets. The following assertions are equivalent:

- (1) The full generating series  $F_{\mathfrak{S}}(t; x, y)$  is D-finite
- (2) the excursions generating series  $F_{\mathfrak{S}}(t; 0, 0)$  is D-finite
- (3) the excursions sequence  $[t^n] F_{\mathfrak{S}}(t; 0, 0)$  is  $\sim K \cdot \rho^n \cdot n^\alpha$ , with  $\alpha \in \mathbb{Q}$
- (4) the group  $\mathcal{G}_{\mathfrak{S}}$  is finite.

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(1)  $\Rightarrow$  (2)  $F_{\mathfrak{S}}(t; 0, 0)$  is the Hadamard product of  $F_{\mathfrak{S}}(t; x, y)$  and  $\frac{1}{1-t}$ , and Hadamard product preserves D-finiteness [Lipshitz 1988]



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(2)  $\Rightarrow$  (3) and (3)  $\Rightarrow$  (4) This talk

## Main result: algorithmic proof of non-D-finiteness

*Theorem* [B.-Rachel-Salvy, 2012]

Let  $\mathfrak{S}$  be one of the 51 step sets with infinite group  $\mathcal{G}_{\mathfrak{S}}$ , and such that the excursions series  $F_{\mathfrak{S}}(t; 0, 0)$  is not equal to 1.

Then  $F_{\mathfrak{S}}(t; 0, 0)$ , and in particular  $F_{\mathfrak{S}}(t; x, y)$ , are non-D-finite.

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- ▷ *Algorithmic, computer-driven, proof*. Uses Gröbner basis computations, polynomial factorization, cyclotomy testing.
- ▷ Based on *two ingredients*: asymptotics + irrationality.

▷ [Kurkova & Raschel 2012] Alternative proof of  $F_{\mathfrak{S}}(t; x, y)$  is non-D-finite. No human proof yet for  $F_{\mathfrak{S}}(t; 0, 0)$  *non-D-finite*.

## First ingredient: probability meets combinatorics

[Denisov & Wachtel 2011] For very general walks in cones in  $\mathbb{N}^d$ :

$$[t^n] F_{\mathfrak{S}}(t; 0, 0) \sim K \cdot \rho^n \cdot n^{-p-d/2}, \quad \text{where}$$

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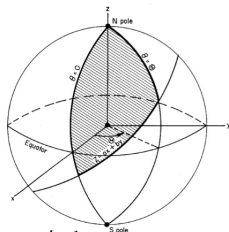
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- ▶  $p$  is  $\sqrt{\lambda + (d/2 - 1)^2} - (d/2 - 1)$ , where  $\lambda > 0$  is the *least Dirichlet eigenvalue of the boundary problem*

$$\begin{cases} \nabla u(x) = -\lambda u(x) & \text{for } x \in \Omega, \\ \text{and } u|_{\partial\Omega} = 0, \end{cases}$$



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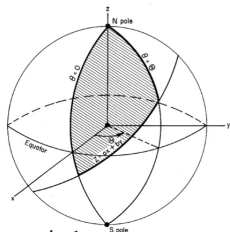
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- ▶ Extends older results for the Brownian motion [Burkholder 1977, DeBlassie 1987, Banuelos & Smits 1997] to walks in discrete time.

## First ingredient: Corollary A

Let  $\mathfrak{S}$  be one of the 51 step sets in  $\mathbb{N}^2$  with infinite group  $\mathcal{G}_{\mathfrak{S}}$ , and such that the excursions series  $F_{\mathfrak{S}}(t; 0, 0)$  is not equal to 1.

Let  $\chi$  be the characteristic polynomial of  $\mathfrak{S}$ . Then the system

$$\frac{\partial \chi}{\partial x} = \frac{\partial \chi}{\partial y} = 0$$

has a unique solution  $(x_0, y_0) \in \mathbb{R}_{>0}^2$ , and

$$[t^n] F_{\mathfrak{S}}(t; 0, 0) \sim K \cdot \rho^n \cdot n^\alpha,$$

where

- ▶  $\rho$  is the algebraic number  $\rho = \chi(x_0, y_0)$ ,
- ▶  $\alpha = \pi / \arccos(c) - 1$ , where  $c < 0$  is the algebraic number

$$c = \frac{\frac{\partial^2 \chi}{\partial x \partial y}}{\sqrt{\frac{\partial^2 \chi}{\partial x^2} \cdot \frac{\partial^2 \chi}{\partial y^2}}}(x_0, y_0).$$

- ▶ Proof uses explicit resolution of the 2D boundary problem.



## Second ingredient: number theory meets combinatorics

### Corollary B

Let  $(a_n)_{n \geq 0}$  be an integer-valued sequence whose  $n$ -th term  $a_n$  behaves asymptotically like  $K \cdot \rho^n \cdot n^\alpha$ , for some real  $K > 0$ . If the *growth constant*  $\rho$  is **transcendental**, or if the *singular exponent*  $\alpha$  is **irrational**, then the generating series  $\sum_{n \geq 0} a_n t^n$  is not D-finite.

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- ▷ If the assumption  $\alpha$  irrational were replaced by  $\alpha$  transcendental, the result would follow by [Birkhoff-Trjitzinsky'32, Turrittin'55]
- ▷ Corollary B is a direct consequence of deep arithmetic results about  $G$ -functions [Chudnovsky'85, André'89, Katz'70]

## Algorithmic blending of probability and arithmetics

To conclude the non-D-finiteness of  $F_{\mathfrak{G}}(t; 0, 0)$ , it suffices to prove that  $\arccos(c)/\pi$  is irrational, where

$$c = \frac{\frac{\partial^2 \chi}{\partial x \partial y}}{\sqrt{\frac{\partial^2 \chi}{\partial x^2} \cdot \frac{\partial^2 \chi}{\partial y^2}}}(x_0, y_0), \quad \text{with} \quad \frac{\partial \chi}{\partial x}(x_0, y_0) = \frac{\partial \chi}{\partial y}(x_0, y_0) = 0,$$

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This can be done *algorithmically* in two main steps:

(S1) determine the minimal polynomial,  $\mu_c$ , of  $c$

(S2) prove that the numerator of  $\mu_c \left( \frac{x^2+1}{2x} \right)$  contains no cyclotomic polynomial factor.

## Step (S1)

- ▷ An algebraic equation for the **exponential growth**  $\rho$  is obtained by eliminating  $x$  and  $y$  from

$$\frac{\partial \chi}{\partial x} = 0, \quad \frac{\partial \chi}{\partial y} = 0, \quad \rho - \chi = 0.$$

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- ▶ This elimination is a routine task in effective algebraic geometry.
- ▶ It is performed using **Gröbner bases** and **polynomial factorization**.
- ▶  $\rho = \chi(x_0, y_0) = \frac{P}{Q}(x_0, y_0)$  is a root of a generator of  $I \cap \mathbb{Q}[t]$ , for

$$I = \left\langle \text{numer} \left( \frac{\partial \chi}{\partial x} \right), \quad \text{numer} \left( \frac{\partial \chi}{\partial y} \right), \quad P(x, y) - t \cdot Q(x, y) \right\rangle.$$

- ▶ Generator obtained by a Gröbner basis computation for  $I$ , using an elimination order with  $(x, y) > t$ .

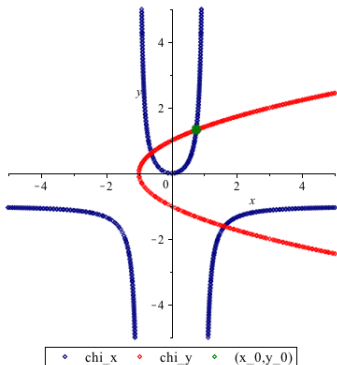


## Step (S1)

▷ Similarly, an algebraic equation for the **correlation coefficient  $c$**  is obtained by eliminating  $x$  and  $y$  from

$$\frac{\partial \chi}{\partial x} = 0, \quad \frac{\partial \chi}{\partial y} = 0, \quad c^2 - \frac{\left(\frac{\partial^2 \chi}{\partial x \partial y}\right)^2}{\frac{\partial^2 \chi}{\partial x^2} \cdot \frac{\partial^2 \chi}{\partial y^2}} = 0.$$

→ the minimal polynomial  $\mu_c$  of  $c$



## Step (S2): $\arccos(c)/\pi$ is irrational

$\arccos(c)/\pi$  is rational  $\iff c = (x + 1/x)/2$  with  $x$  a root of unity

$\iff$

$R(x) = x^{\deg \mu_c} \cdot \mu_c\left(\frac{x^2+1}{2x}\right)$  is divisible by a cyclotomic polynomial

**Our algorithm:** In all the 51 cases,  $R(x)$  is irreducible, it has degree  $\leq 28$ , and at least one coefficient of absolute value greater than 3.

[Lehmer'36, Rosser & Schoenfeld'62]: if a cyclotomic polynomial has degree  $\leq 30$ , then its coefficients belong to  $\{-2, -1, 0, 1, 2\}$

▷ In the 51 cases,  $\arccos(c)/\pi$ , and thus also  $\alpha$ , is irrational, q.e.d.

## The algorithm on the example



```
> S:=[[-1,0],[0,1],[1,0],[1,-1],[0,-1]]:  
> chi:=add(x^s[1]*y^s[2],s=S);
```

$$\chi := \frac{1}{x} + \frac{1}{y} + x + y + \frac{x}{y}$$

```
> chi_x:=numer(diff(chi,x));chi_y:=numer(diff(chi,y));
```

$$\chi_x := x^2 + x^2y - y, \quad \chi_y := y^2 - x - 1.$$

```
> G:=Groebner[Basis]([chi_x,chi_y, numer(t^2-  
diff(chi,x,y)^2/diff(chi,x,x)/diff(chi,y,y))],lexdeg([x,y],[t]));
```

```
> p:=factor(op(remove(has,G,{x,y})));
```

$$p := (4t^2 + 1)(8t^3 + 8t^2 + 6t + 1)(8t^3 - 8t^2 + 6t - 1).$$

*This polynomial has only two real roots,  $\pm c$ . The sign condition  $c < 0$  identifies the minimal polynomial of  $c$  as  $\mu_c = 8t^3 + 8t^2 + 6t + 1$*

```
mu_c:=8*t^3+8*t^2+6*t+1:
```

```
R:=expand(x^3*subs(t=(x^2+1)/x/2, mu_c),sort);
```

$$R(x) = x^6 + 2x^5 + 6x^4 + 5x^3 + 6x^2 + 2x + 1.$$

```
> irreduc(R), numtheory[isyclotomic](R,x);
```

true, false

## Summary and future work

- ☺ **algorithmic proof** of non-D-finiteness of  $F(t; 0, 0)$  and  $F(t; x, y)$  for the 51 non-singular cases with infinite group
- ☺ 2D classification of  $F(t; 0, 0)$  and  $F(t; x, y)$  is **fully completed**

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### *Extensions*

1. **Longer 2D steps** [B., Bousquet-Mélou & Melczer, in progress]
  - 680 step sets with one large step, 643 proven non D-finite
  - 5910 step sets with two large steps, 5754 proven non D-finite
2. **3D walks** [B., Raschel, Salvy, in progress]
  - New phenomenon (empirically discovered, no proof yet):  $\exists$  step sets (3D Kreweras) **with finite group and non-D-finite GF!**?

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### Open questions

1. Prove directly the needed arithmetic and combinatorial results.
2. Are the singular exponents  $\alpha$  even transcendental?
3. Nature of  $F(t; 1, 1)$  in the 51 non-singular cases?

Thanks for your attention!